Emergence of integer quantum Hall effect from chaos in the kicked rotor

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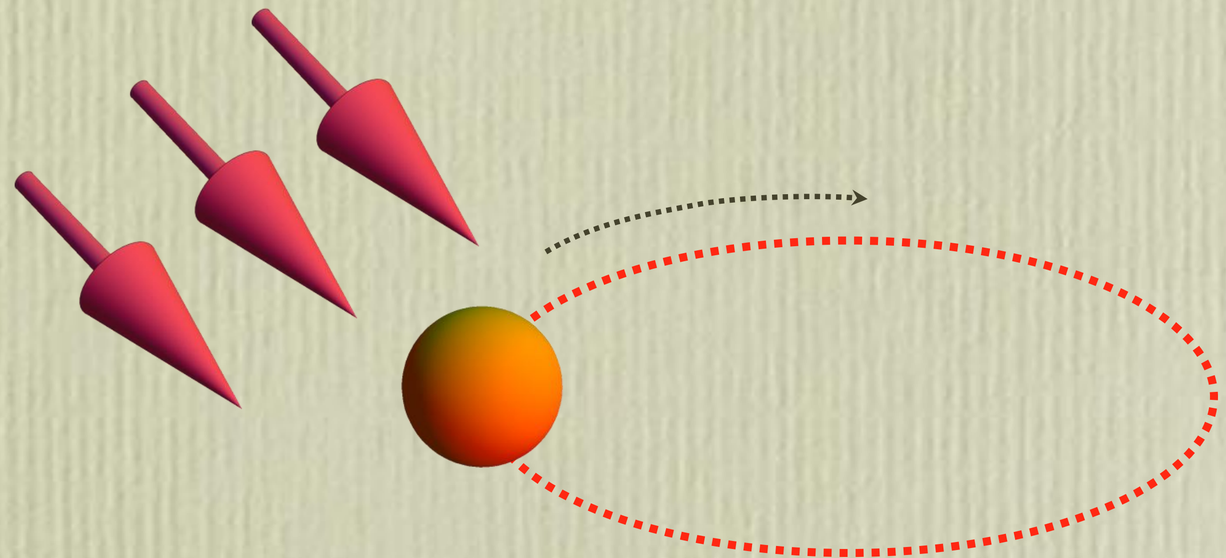
Y. Chen and C. Tian, Phys. Rev. Lett. 113, 216802 (2014)
C. Tian, Y. Chen, and J. Wang, Phys. Rev. B 93, 075403 (2016) (38 pages)

Outline

- A brief introduction to kicked rotor
- Planck’s quantum-driven IQHE in kicked spin rotor — phenomenon, analytic theory, and numerical confirmation
- Conclusion

What is kicked rotor?
- a free rotating particle under the influence of sequential time-periodic driving



$$\hat{H} = \frac{l^2}{2I} + K\cos\theta \sum_m \delta(t - mT)$$

$\hbar T/I \to h_e$ — Planck's quantum (effective Planck's constant)

$KT/I \to K$

$lT/I \to l$

$t/T \to t$

$$\hat{l} = h_e \hat{n}$$

controlled by two parameters:

- Planck's quantum h_e
- nonlinear parameter K

$$E(t) = \frac{1}{2}\langle \psi(t)|\hat{n}^2|\psi(t)\rangle$$

Chirikov standard map
- the birth of classical kicked rotor ($h_e \to 0$)

NUCLEAR PHYSICS INSTITUTE OF THE SIBERIAN
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Report 267

<u>RESEARCH CONCERNING THE THEORY OF</u>
<u>NON-LINEAR RESONANCE AND STOCHASTICITY</u>

B.V. Chirikov

Novosibirsk, 1969

Chirikov standard map
- the birth of classical kicked rotor ($h_e \to 0$)

or

$$\omega_{n+1} = \omega_n + \varepsilon \cdot \cos 2\pi \psi_n$$
$$\psi_{n+1} = \left\{ \psi_n + \frac{T}{2\pi} \omega_{n+1} \right\} \qquad (2.1.15)$$

Here the curly brackets represent the fractional part of the argument -- a convenient way of specifying the periodic dependence. The coefficients of the model equations (2.1.14) and (2.1.15) are selected so that the Jacobian $|\partial(\omega_{n+1}, \psi_{n+1})/\partial(\omega_n, \psi_n)| = 1$ exactly. The reasons for the choice of two forms of dependence on ψ will be clear from what follows (see Section 2.4).

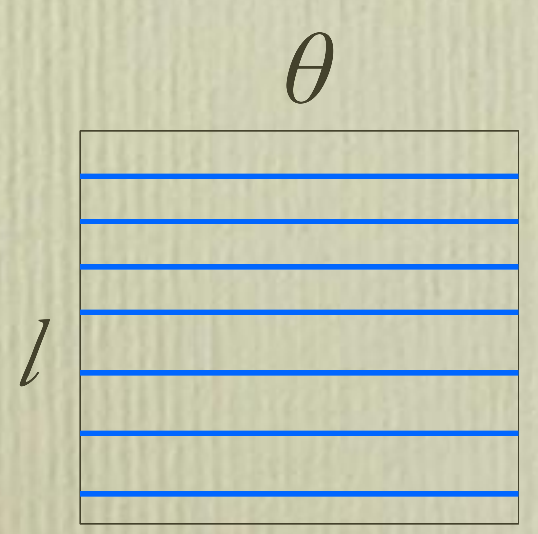
We chose for our basic model (2.1.11) a perturbation in the form of short kicks, essentially in the form of a δ-function. This choice is not very special or exceptional; on the contrary, it is typical, since the sum in the right-hand part (2.1.6), when there are a large number of terms, actually represents either a short kick (or series of kicks) or frequency-modulated perturbation. In the latter case periodic crossing of the resonance takes place, which according to the results of Section 1.5 is also equivalent to some kick [(1.5.7) and (1.5.9)]. Thus it can be expected that the properties of model (2.1.11) will be in a sense typical for the problem of the interaction of the resonances and stochasticity.

classical kicked rotor

$$l_{n+1} = l_n + K \sin\theta_{n+1}$$
$$\theta_{n+1} = \theta_n + l_n$$

$\xrightarrow{K=0}$

$$l_{n+1} = l_n$$
$$\theta_{n+1} = \theta_n + l_n$$



Liouville integrability:
- regular foliation of phase space
- action variables = complete sets of invariants of Hamiltonian flow

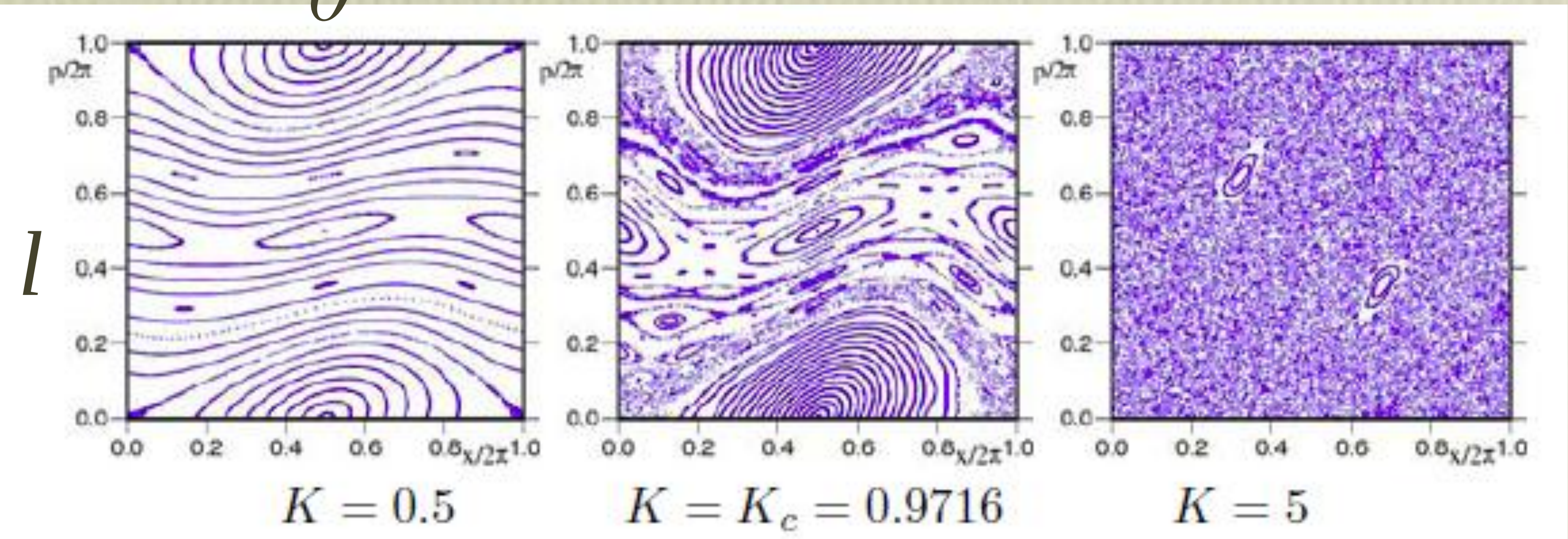
classical kicked rotor

$$l_{n+1} = l_n + K \sin \theta_{n+1}$$
$$\theta_{n+1} = \theta_n + l_n$$

K increases from 0.

θ

l



from B. Chirikov and D. Shepelyansky, Scholarpedia 3, 3550 (2008)

transition from weak chaoticity (KAM, nearly integrable) to strong chaoticity

classical kicked rotor

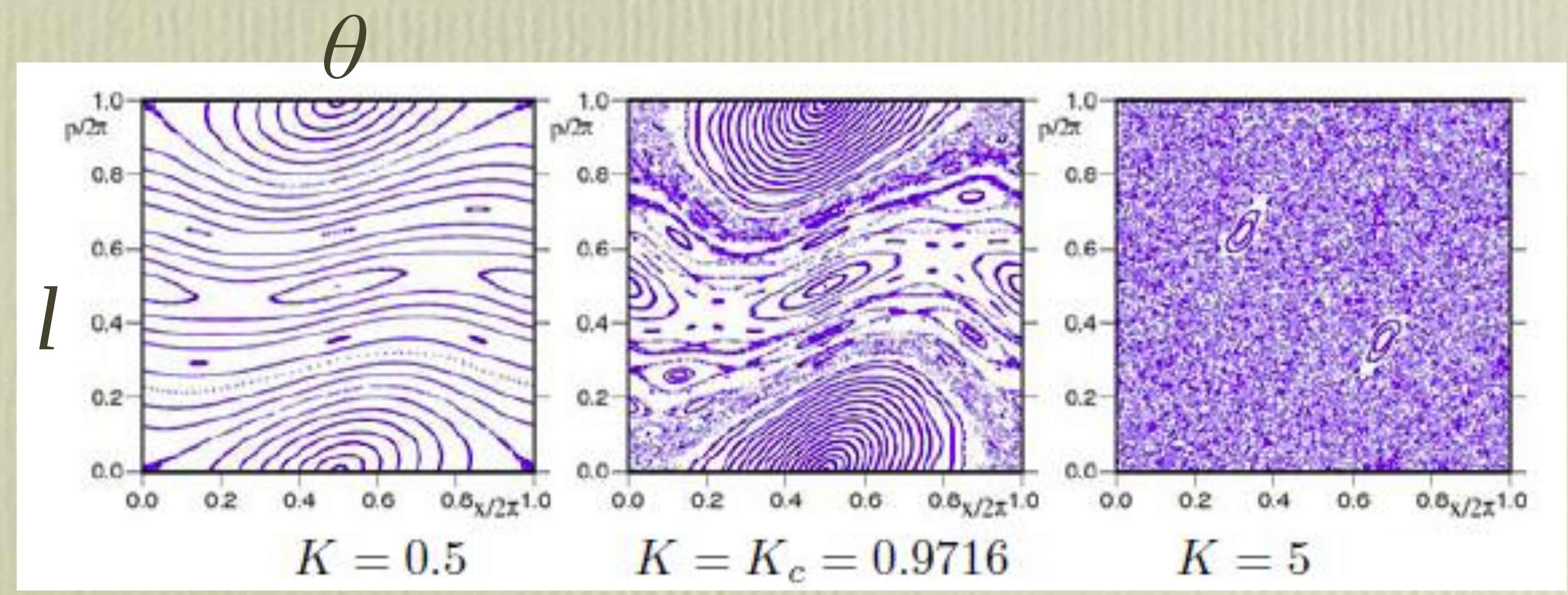
$$l_{n+1} = l_n + K \sin \theta_{n+1}$$

$$\theta_{n+1} = \theta_n + l_n$$

large and general K : lose memory on θ

random walk - Brownian motion - in l - space

$$\frac{E(t)}{t} = const.$$

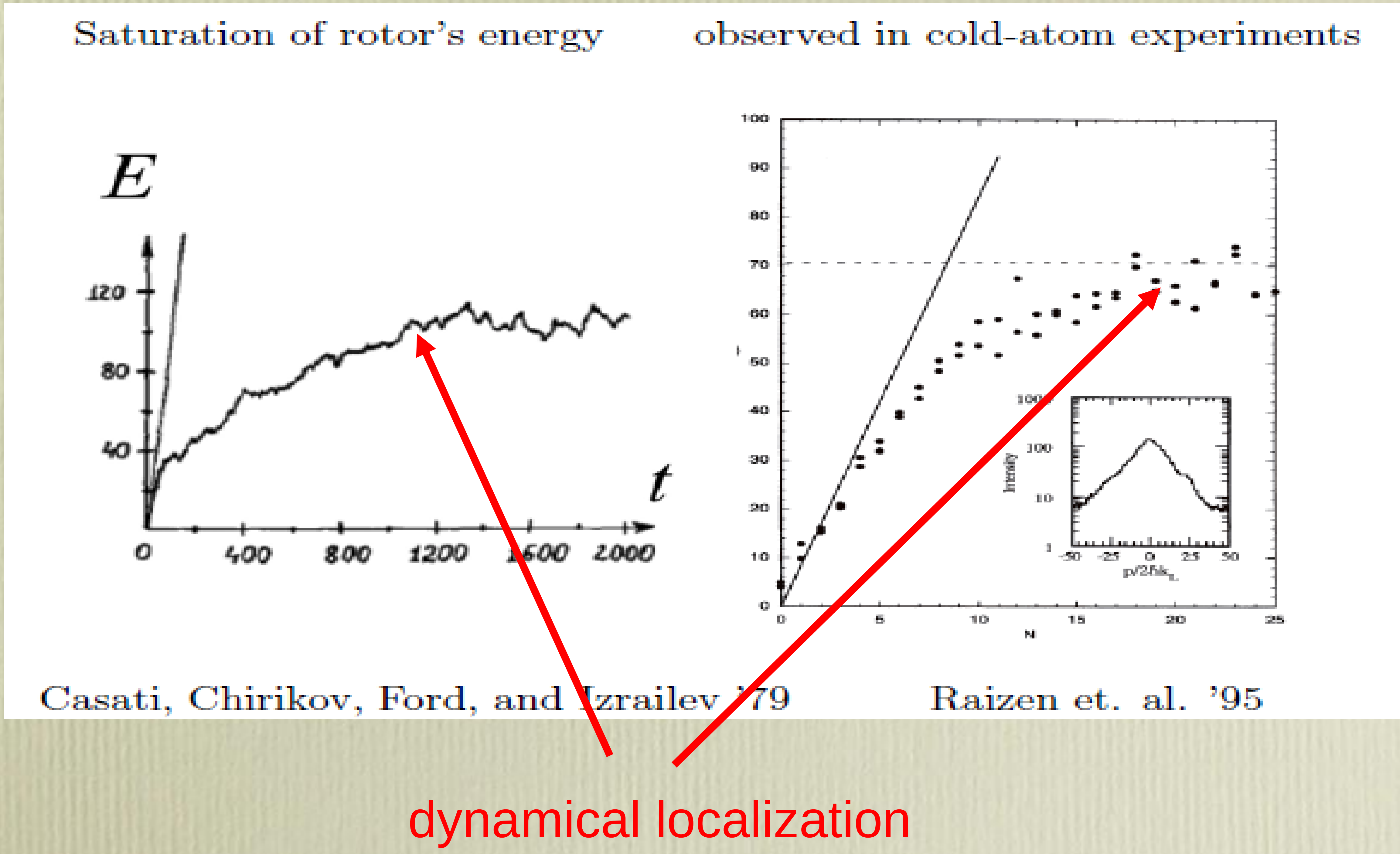


from B. Chirikov and D. Shepelyansky, Scholarpedia 3, 3550 (2008)

transition from "insulator" (confined motion in l space) to "classical normal metal" (deconfined motion in l space)

What happens to quantum kicked rotor ($h_e \rangle 0$)?

(generic) irrational values of $h_e/(4\pi)$



PHYSICAL REVIEW LETTERS

VOLUME 49 — 23 AUGUST 1982 — NUMBER 8

Chaos, Quantum Recurrences, and Anderson Localization

Shmuel Fishman, D. R. Grempel, and R. E. Prange
Department of Physics and Center for Theoretical Physics, University of Maryland, College Park, Maryland 20742
(Received 4 April 1982)

Bloch-Floquet theory

dynamical-Anderson localization analogy

$$\hat{U}|\phi_\alpha\rangle = e^{i\omega_\alpha}|\phi_\alpha\rangle$$
$$\hat{U}|\phi_\alpha(t)\rangle = e^{i\omega_\alpha t}|\phi_\alpha(t)\rangle$$
$$|\phi_\alpha(t+1)\rangle = |\phi_\alpha(t)\rangle$$
$$\phi_\alpha(n) = \langle n|\phi_\alpha\rangle \quad w_n = |\phi_\alpha(n)|$$

$$\bar{\phi}_\alpha(n) = \tfrac{1}{2}(\langle n\tilde{h}|\phi_\alpha^+\rangle + \langle n\tilde{h}|\phi_\alpha^-\rangle)$$
$$|\phi_\alpha^+\rangle = e^{iK\cos\hat{\theta}/\tilde{h}}|\phi_\alpha^-\rangle$$
$$\tan(\omega - \tilde{h}n^2/2)\bar{\phi}_\alpha(n) + \sum_r W_{n-r}\bar{\phi}_\alpha(r) = 0$$
$$\hat{W} = -\tan(K\cos\hat{\theta}/2\tilde{h})$$
W_n rapidly decays away at $|n| > K/\tilde{h}$

pseudo-randomness at irrational $\tilde{h}/(4\pi)$

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$$\hat{W} = -\tan(K\cos\hat{\theta}/2\tilde{h})$$
$$W_n \text{ rapidly decays away at } |n| > K/\tilde{h}$$

quantum kicked rotor
= quantum disordered system?

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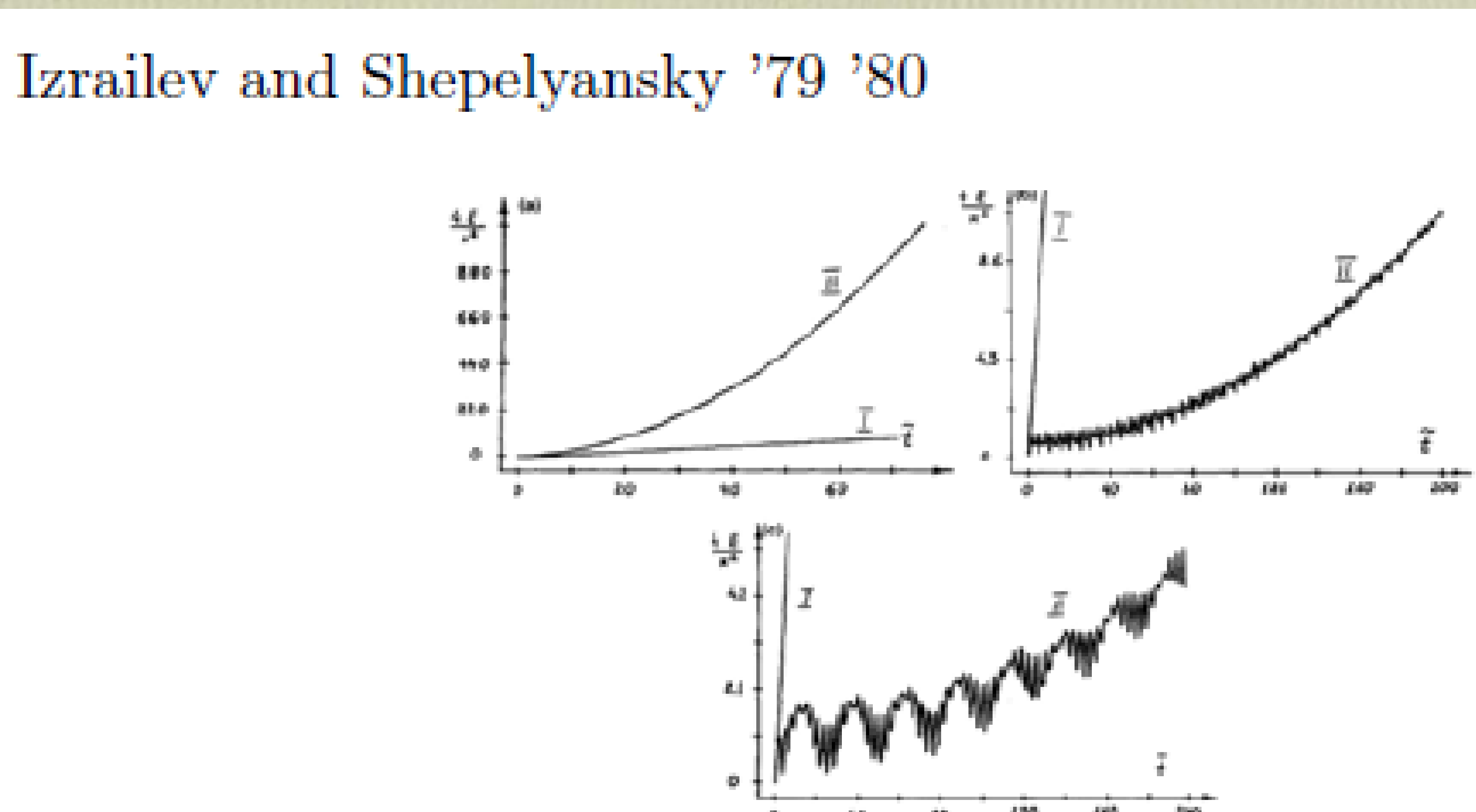
$$\hat{W} = -\tan(K\cos\hat{\theta}/2\tilde{h})$$
$$W_n \text{ rapidly decays away at } |n| > K/\tilde{h}$$

NO!

sensitivity to the value of $h_e/(4\pi)$:

$h_e/(4\pi) = p/q$

small q: nonuniversal



$$E(t) \sim t^2$$

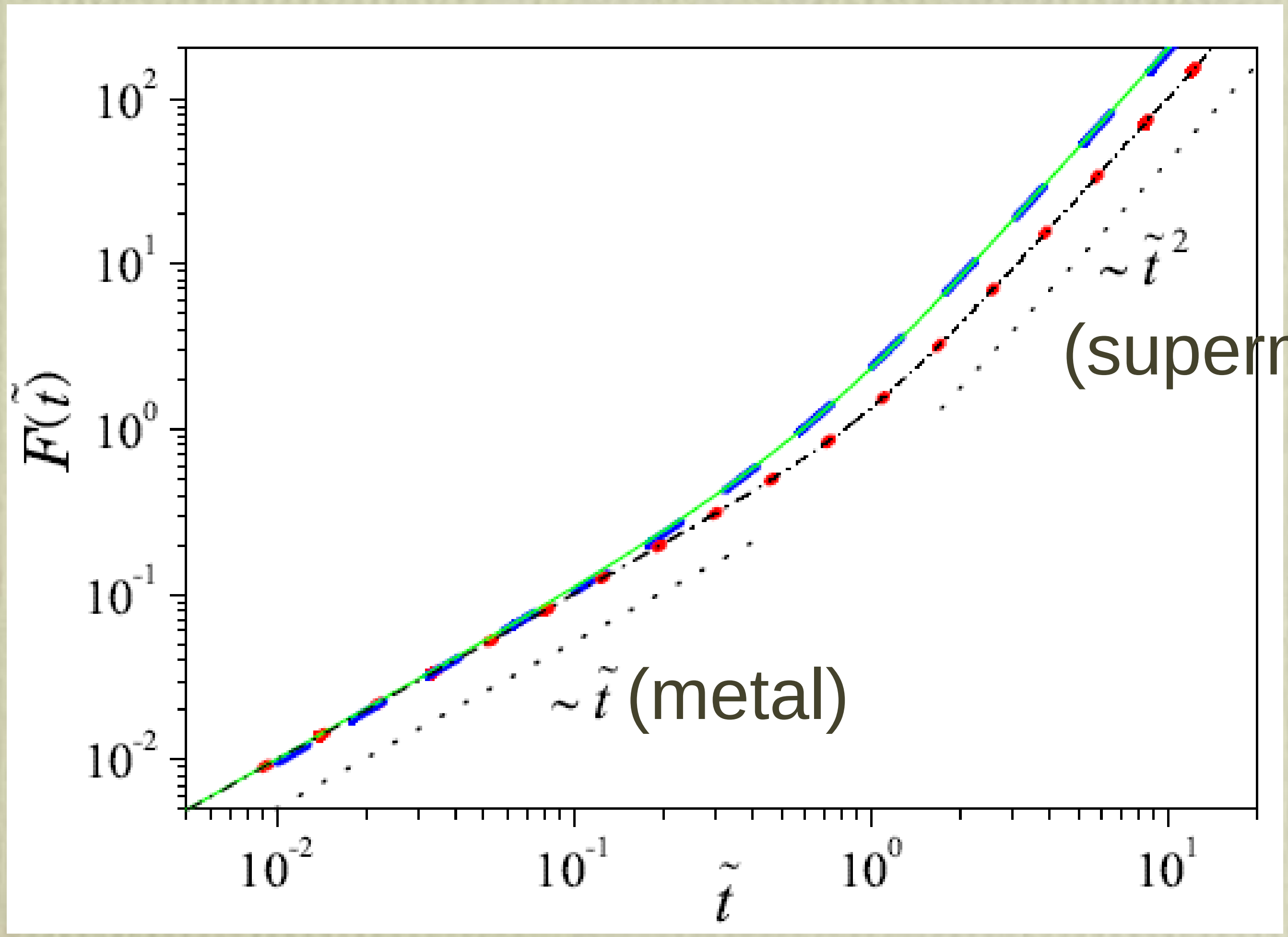
supermetal

result of translation symmetry: $n \rightarrow n+q$

sensitivity to the value of $h_e/(4\pi)$:

$h_e/(4\pi) = p/q$

large q: universal metal-supermetal dynamics crossover



$$\hat{H} = H_0(\hat{n}) + K \cos\hat{\theta} \sum_m \delta(t-m).$$

$$H_0(\hat{n}) = \sum_{k=0}^{\infty} c_k \hat{n}^k,$$

$$\hat{T}_c: \quad \hat{n} \to \hat{n},\ \hat{\theta} \to -\hat{\theta},\ t \to -t.$$ √

$$\hat{T}_i: \quad \hat{n} \to -\hat{n}.$$???

Fang, Tian, and Wang, PRB '15

sensitivity to the value of $h_e/(4\pi)$:

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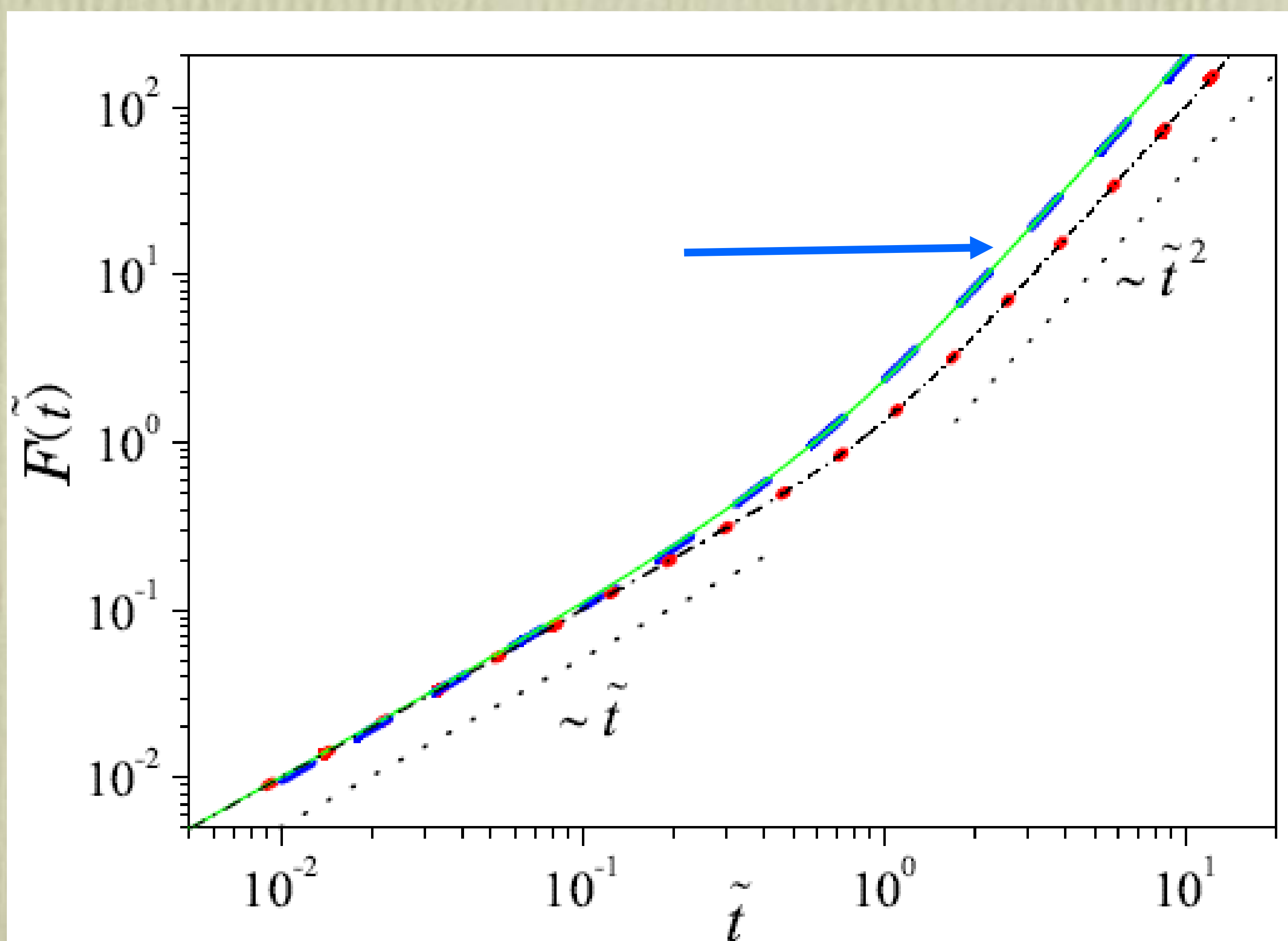

$$\hat{T}_i : \quad \hat{n} \to -\hat{n}.$$ √

→ orthogonal symmetry

$$F(\tilde{t}) = \frac{1}{8}\int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2 \int_{-1}^{1} d\lambda \delta(2\tilde{t} + \lambda - \lambda_1\lambda_2) \times \frac{(1-\lambda^2)(1-\lambda^2-\lambda_1^2-\lambda_2^2+2\lambda_1^2\lambda_2^2)^2}{(\lambda^2+\lambda_1^2+\lambda_2^2-2\lambda\lambda_1\lambda_2-1)^2}. \quad (23)$$

- derived by field theory
- confirmed by random matrix theory
- confirmed by numerical experiments

Fang, Tian, and Wang, PRB ‘15

sensitivity to the value of $h_e/(4\pi)$:

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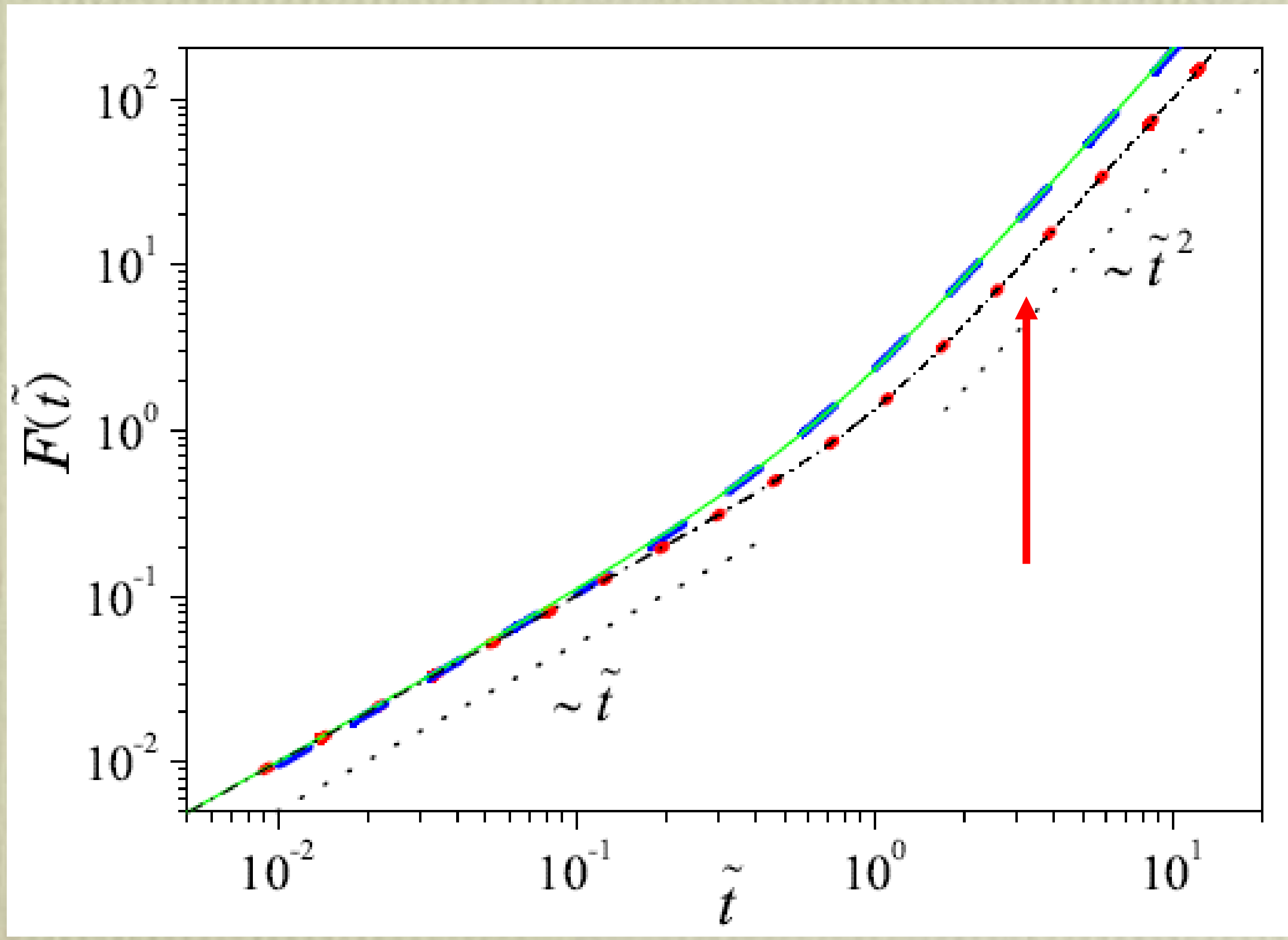
large q: universal metal-supermetal dynamics crossover



$\hat{T}_i: \quad \hat{n} \to -\hat{n}.$ x

→ unitary symmetry

$$F(\tilde{t}) = \begin{cases} \tilde{t} + \frac{1}{3}\tilde{t}^3, & 0 < \tilde{t} < 1, \\ \tilde{t}^2 + \frac{1}{3}, & \tilde{t} > 1. \end{cases}$$

- derived by field theory
- confirmed by random matrix theory
- confirmed by numerical experiments

Fang, Tian, and Wang, PRB '15

quasiperiodically quantum kicked rotor: irrational $h_e/(4\pi)$

$$K \to K(\tilde{\omega} t)$$

driven by *d*-incommensurate frequencies
Idea dated back to Casati, Guarneri, and Shepelyansky, PRL '89
Experiment: Delande, Garreau et.al., PRL '08, 09
Field theory: Tian, Altland, and Garst PRL 11'

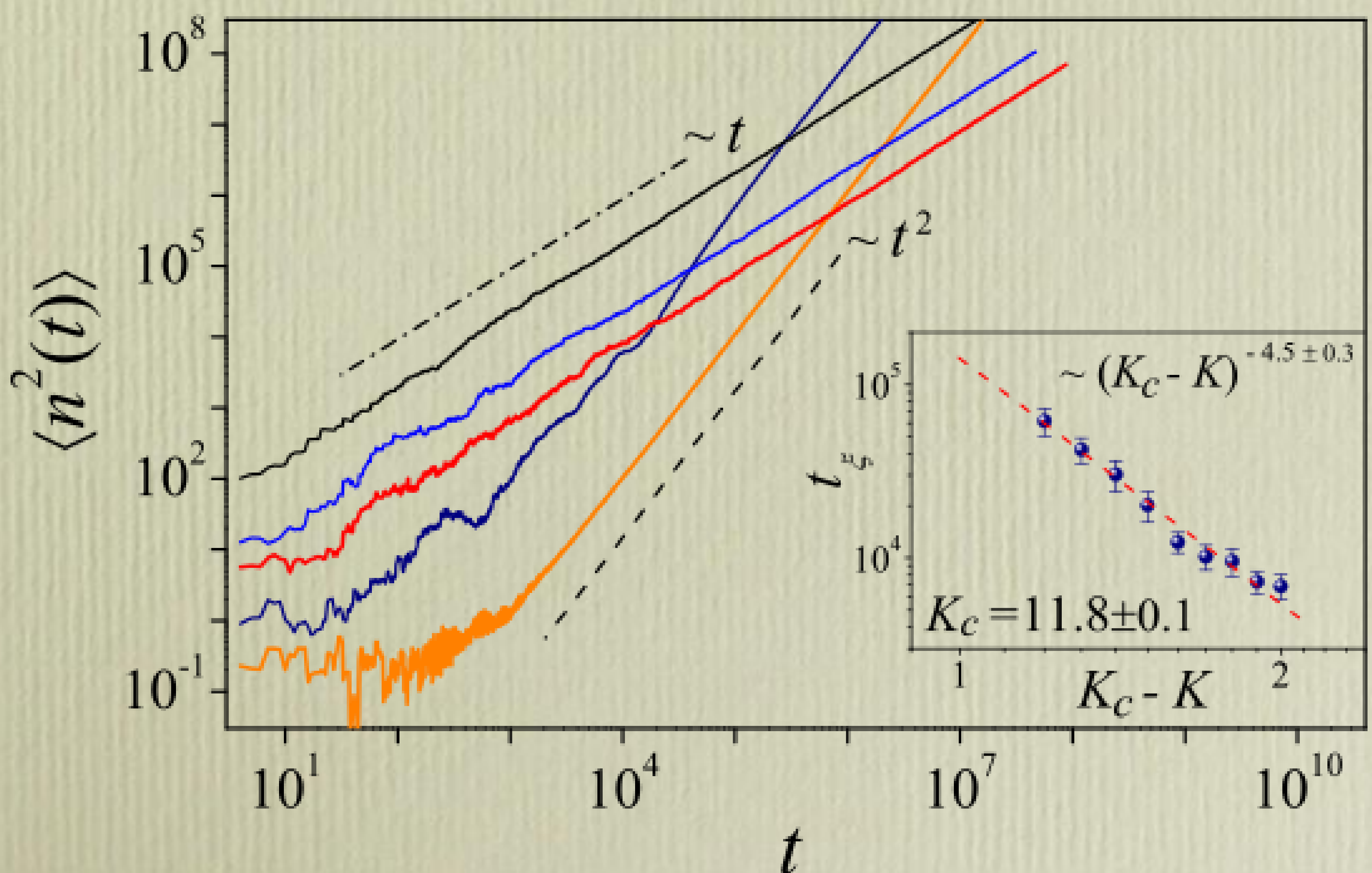
$d \leq 2$: Anderson insulator

$d > 2$: Anderson transition

- $K/\tilde{h} \gg 1$: $E(t) \sim t \Rightarrow$ metallic;
- $K/\tilde{h} = \mathcal{O}(1)$: Diffusion constant D_ω is strongly renormalized $D_\omega \xrightarrow{\omega \to 0} i\omega \to E(t) \to const. \Rightarrow$ insulating;
- At critical point: $D_\omega \sim (-i\omega)^{(d-2)/d} \Rightarrow E(t) \sim t^{2/d}$

quasiperiodically quantum kicked rotor: rational $h_e/(4\pi)$

- Anderson insulator turned into supermetal (E~t^2) ;
- Anderson metal-insulator transition turned into metal-supermetal transition (Tian, Altland, and Garst, PRL '11)



$$t_\xi \sim (K_c - K)^{-\alpha}$$

$$K_c = 11.8 \pm 0.1$$

$$\alpha = 4.5 \pm 0.3.$$

numerical confirmation (Wang, Tian, Altland, PRB 14')
p=1,q=3,d=4

rich Planck's quantum-driven phenomena in spinless kicked rotors;

associated with the restoration (breaking) of translation symmetry in the angular momentum space.

spinful kicked rotor?

Outline

- A brief introduction to kicked rotor
- Planck's quantum-driven IQHE in kicked spin rotor — phenomenon, analytic theory, and numerical confirmation
- Conclusion

quasiperiodically quantum kicked spin-1/2 rotor

$$\hbar T/I \to h_e$$

$$l_1 T/I \to l_1$$
$$t/T \to t$$
$$\tilde{\omega} T \to \tilde{\omega}$$

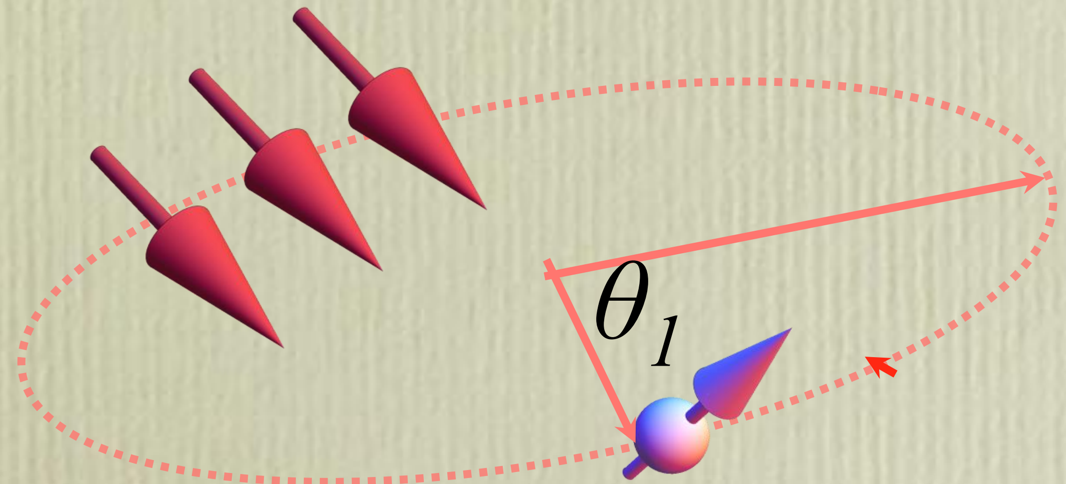
$$\hat{H} = \frac{\hat{l}_1^2}{2} + \sum_m V(\theta_1, \theta_2 + \tilde{\omega} t)\delta(t - m)$$

incommensurate with 2π

$$V(\theta_1, \theta_2 + \tilde{\omega} t) = \sum_{i=1}^{3} V_i(\theta_1, \theta_2 + \tilde{\omega} t)\sigma^i$$

$$\equiv \vec{V} \cdot \vec{\sigma}$$

σ^i : Pauli matrix



quasiperiodically quantum kicked spin-1/2 rotor

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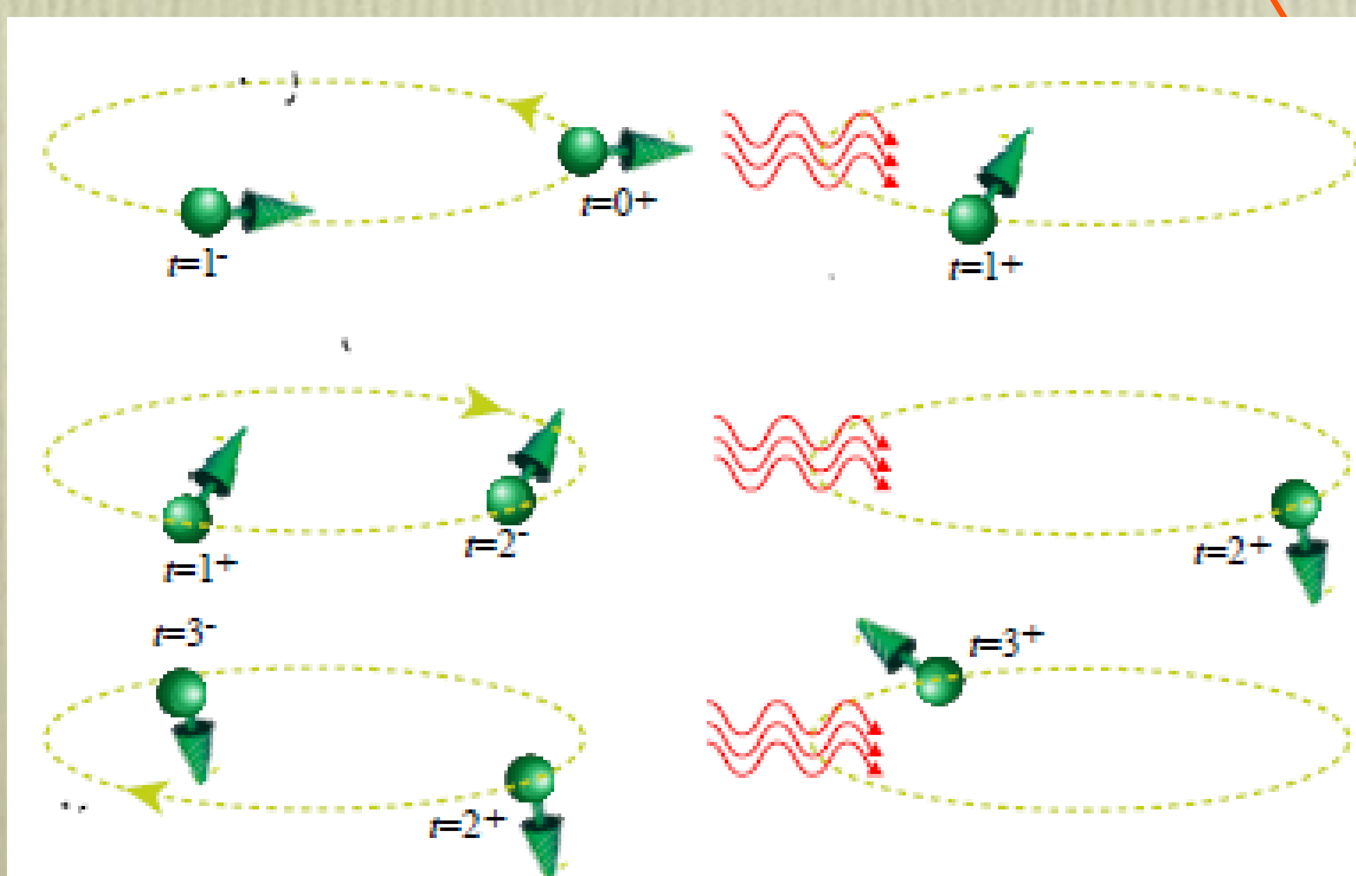

quasiperiodically quantum kicked spin-1/2 rotor

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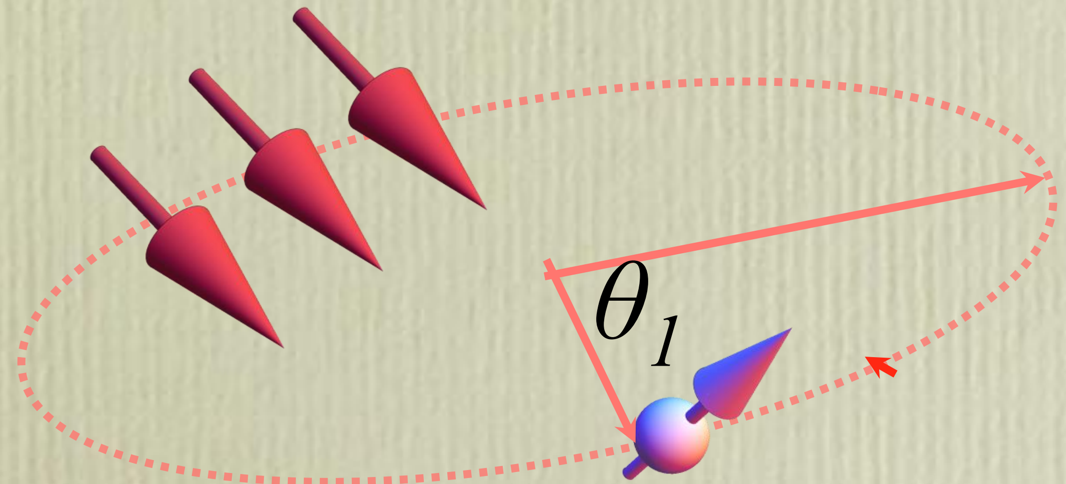


TABLE II. Parities of V_i.

	$V_1(\theta_1, \theta_2)$	$V_2(\theta_1, \theta_2)$	$V_3(\theta_1, \theta_2)$
θ_1	odd	even	even
θ_2	even	odd	even

unitary class

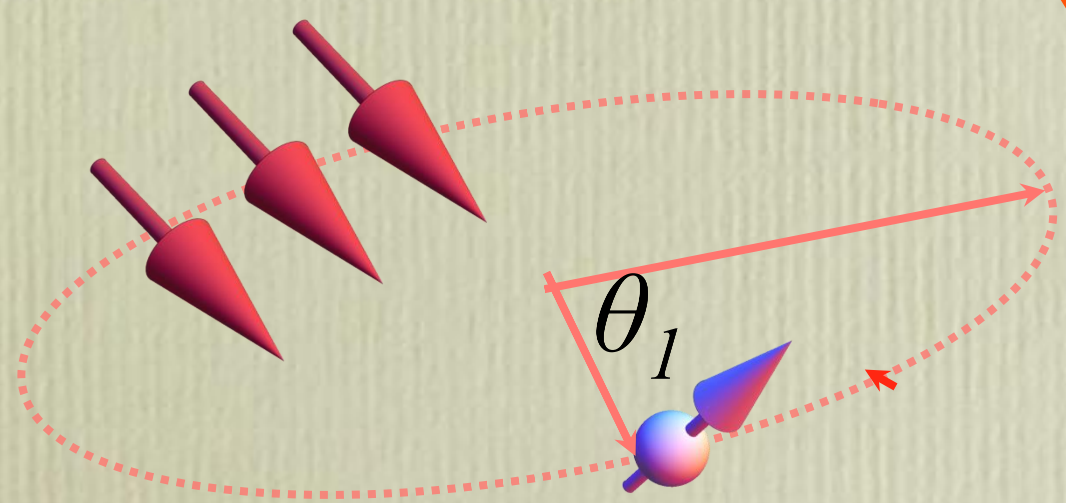
quasiperiodically quantum kicked spin-1/2 rotor

$$\hat{H} = \frac{\hat{l}_1^2}{2} + \sum_m V(\theta_1, \theta_2 + \tilde{\omega} t)\,\delta(t - m)$$

incommensurate with 2π

$$V(\theta_1, \theta_2 + \tilde{\omega} t) = \sum_{i=1}^{3} V_i(\theta_1, \theta_2 + \tilde{\omega} t)\,\sigma^i$$

$$\equiv \vec{V} \cdot \vec{\sigma}$$



Microscopically, the system is controlled by single parameter – Planck's quantum h_e.

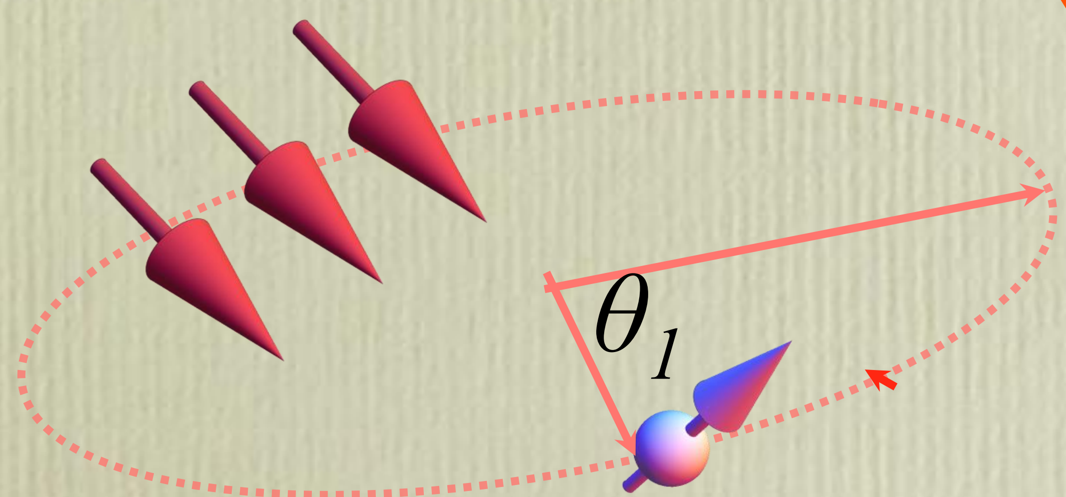
quasiperiodically quantum kicked spin-1/2 rotor

$$\hat{H} = \frac{\hat{l}_1^2}{2} + \sum_m V(\theta_1, \theta_2 + \tilde{\omega} t)\delta(t - m)$$

incommensurate with 2π

$$V(\theta_1, \theta_2 + \tilde{\omega} t) = \sum_{i=1}^{3} V_i(\theta_1, \theta_2 + \tilde{\omega} t)\sigma^i$$

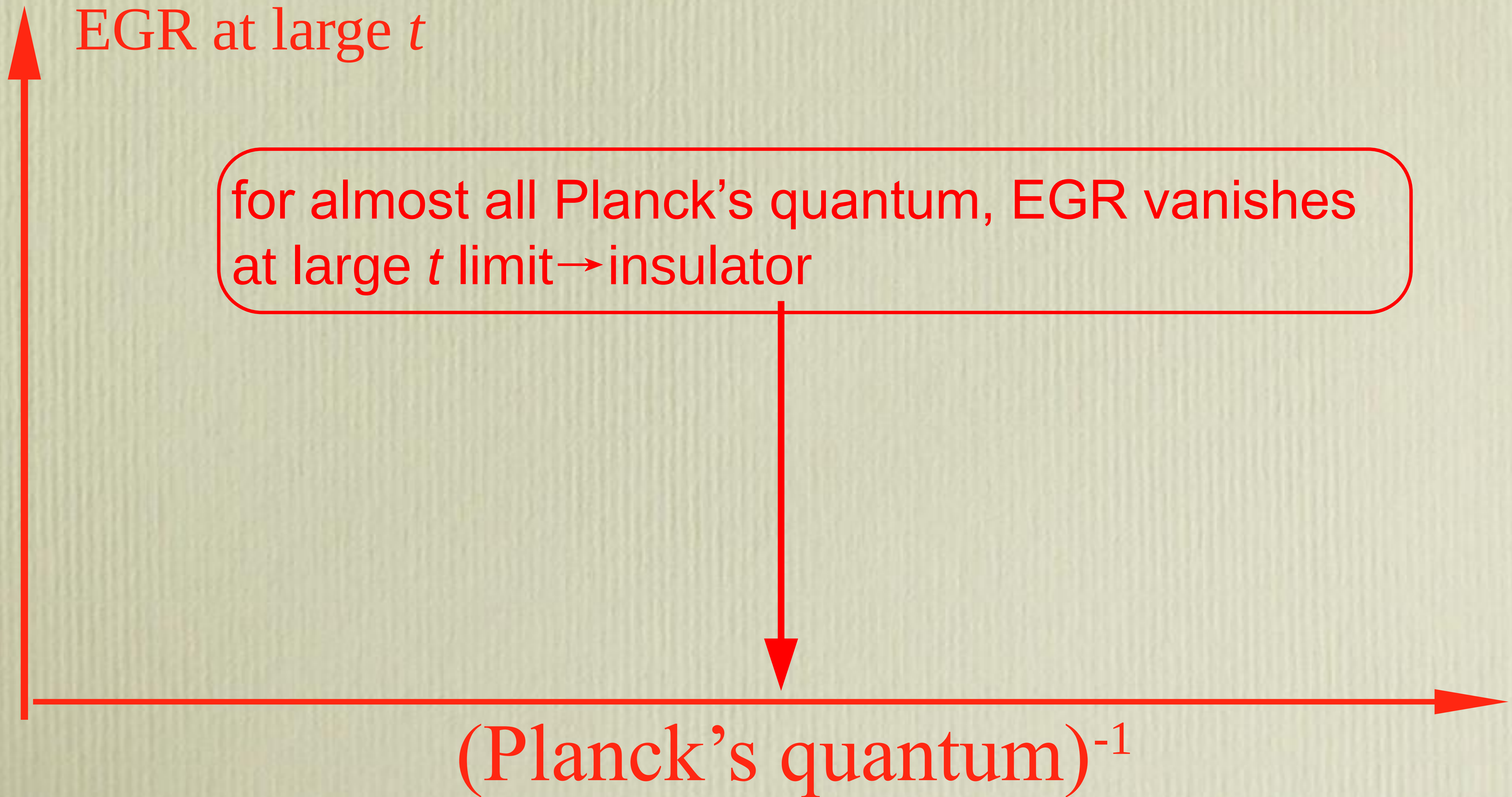
$$\equiv \vec{V} \cdot \vec{\sigma}$$



Macroscopically, the system is controlled by two phase parameters – energy growth rate (EGR) and (hidden or emergent) quantum number.

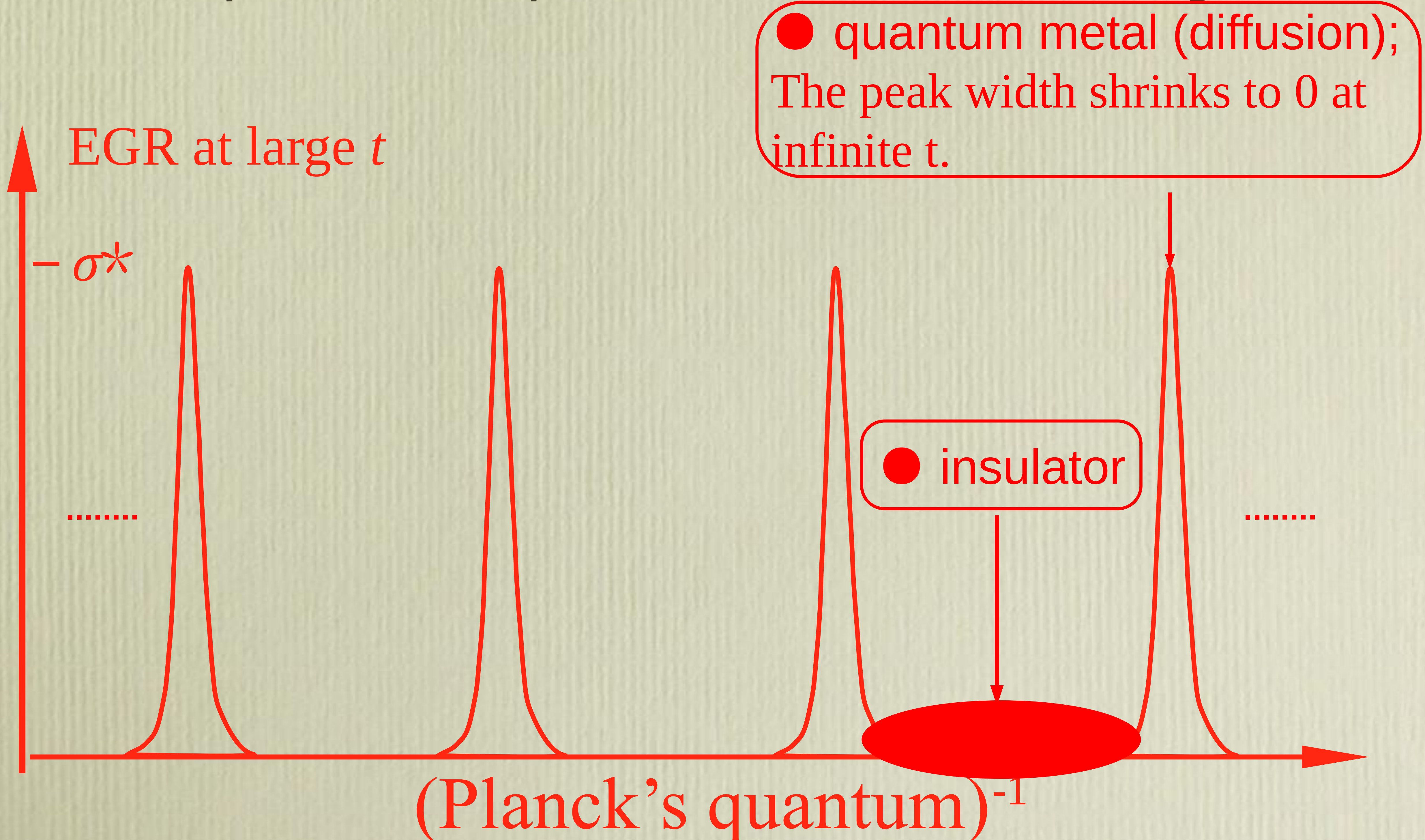
Planck’s quantum-driven IQHE (I)

Planck’s quantum dependence of EGR in long times

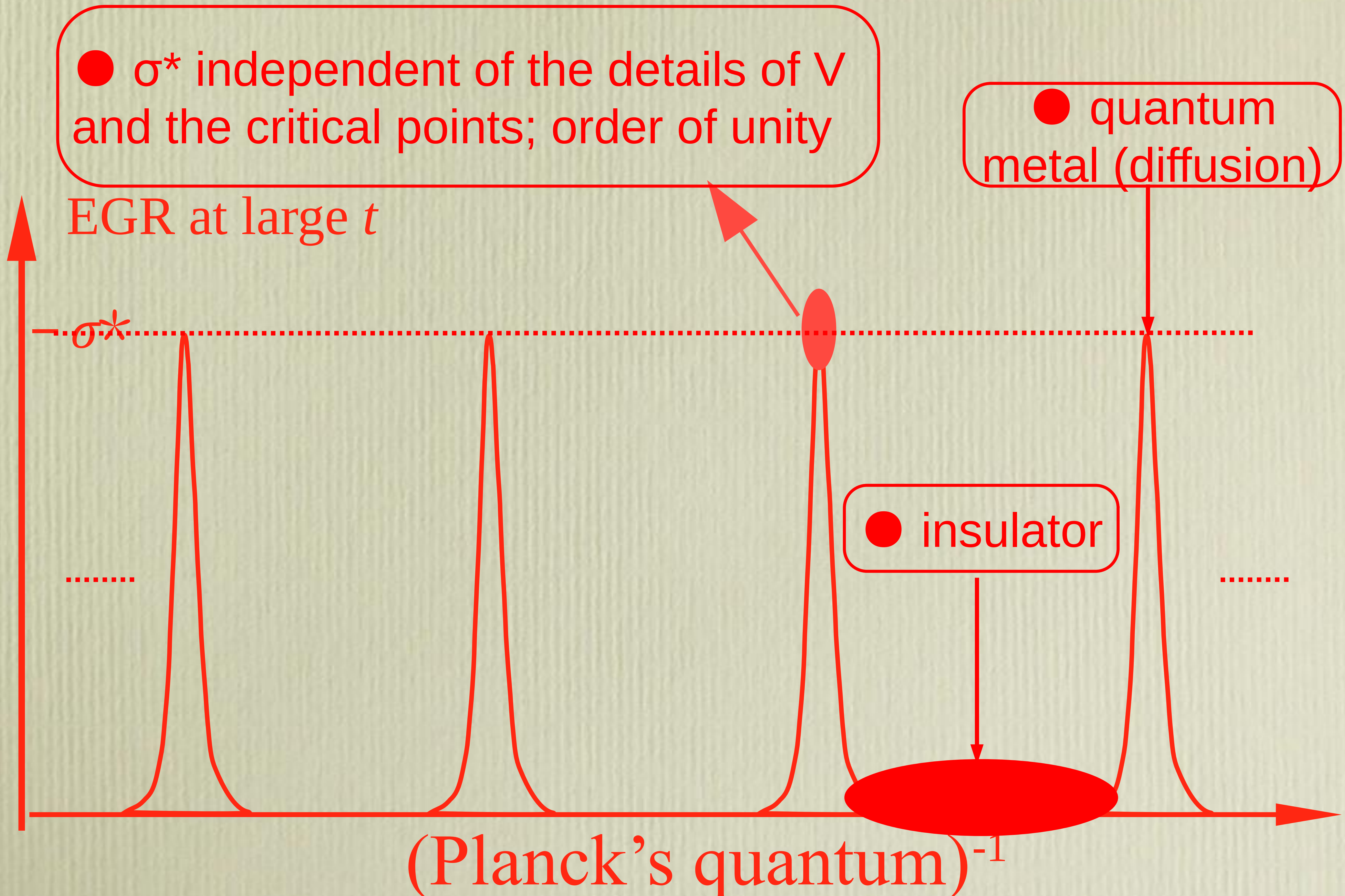


Planck's quantum-driven IQHE (II)

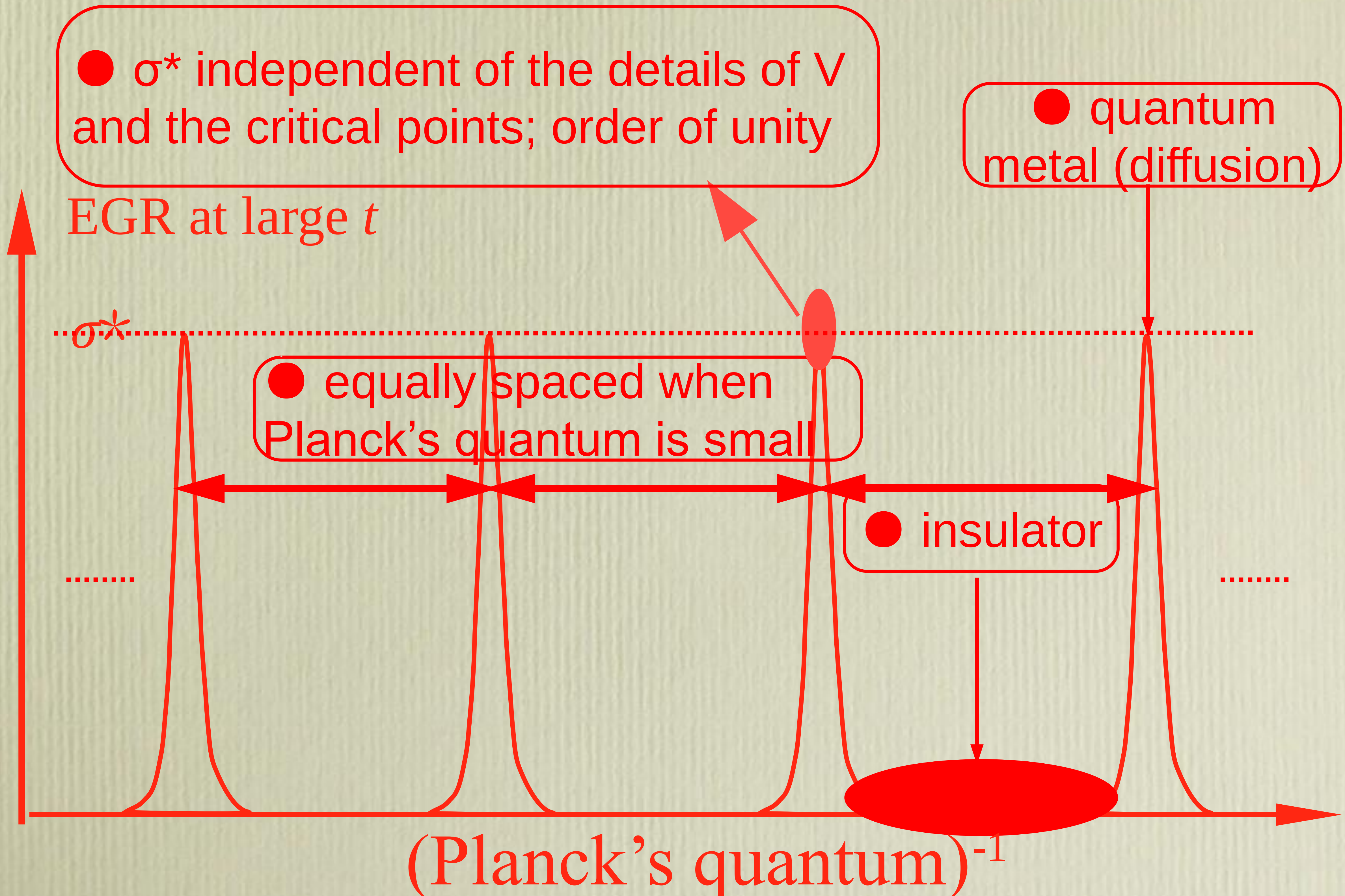
Planck's quantum dependence of EGR in long times



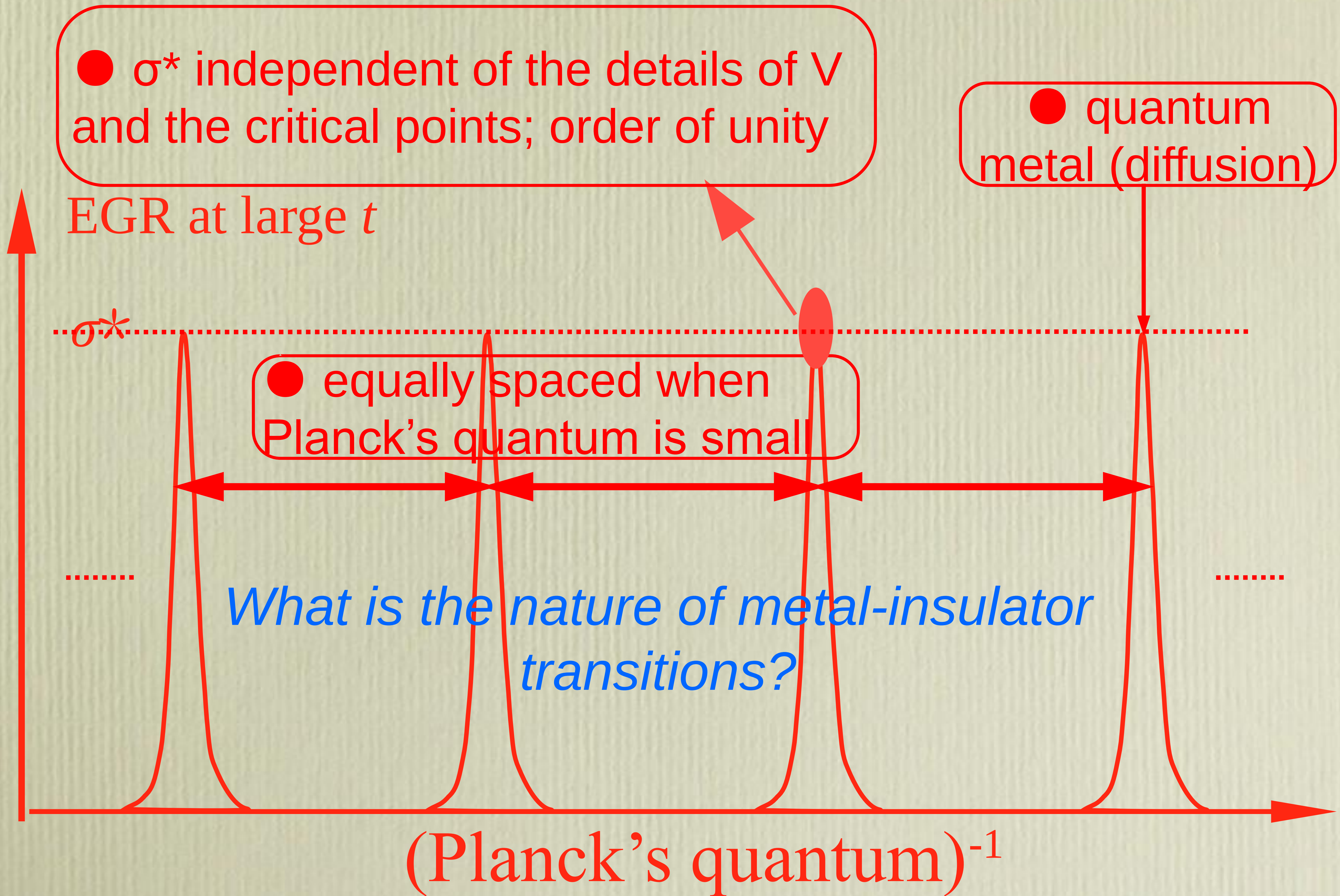
Planck's quantum-driven IQHE(III)



Planck's quantum-driven IQHE (IV)

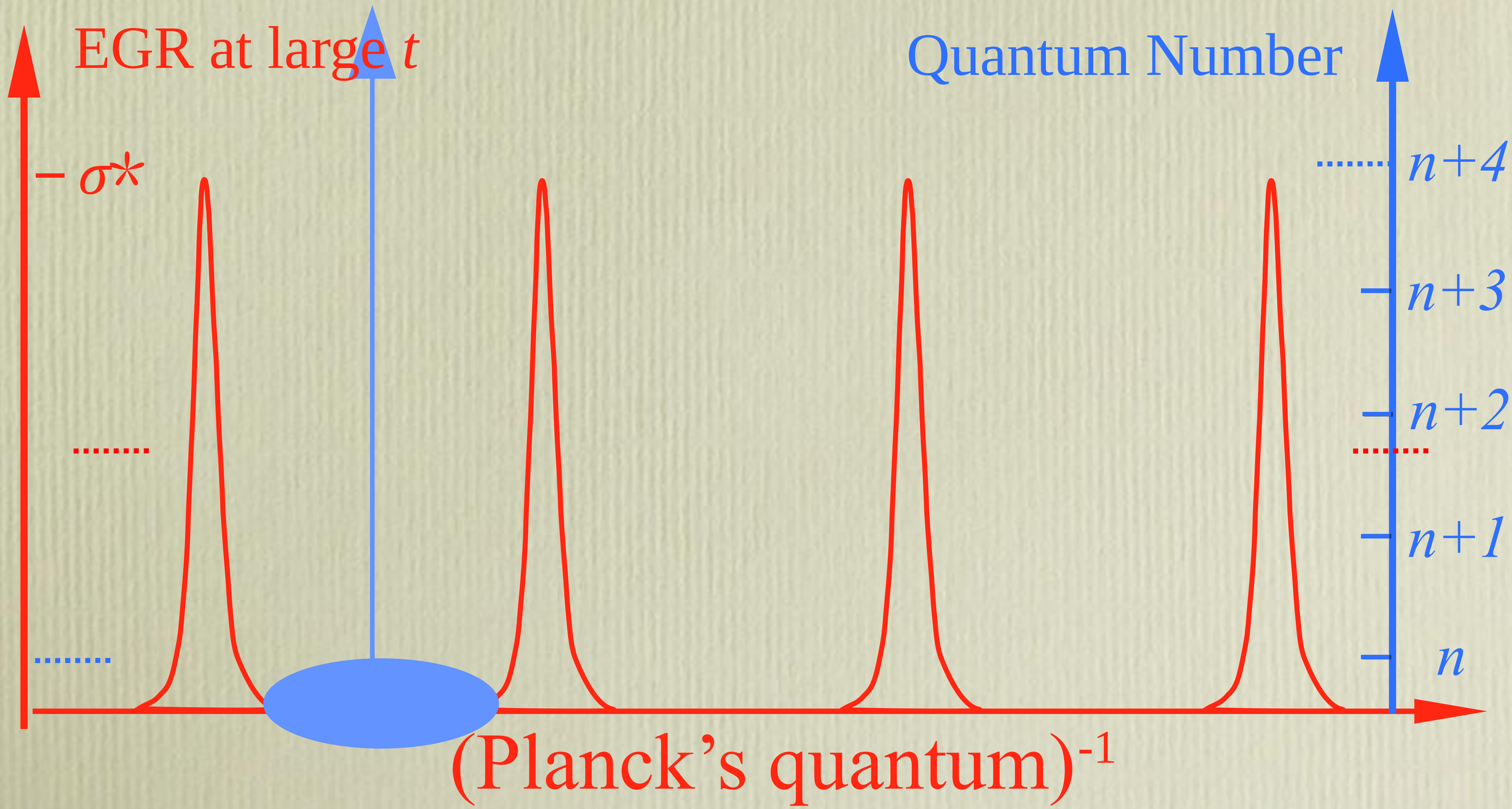


Planck's quantum-driven IQHE (V)



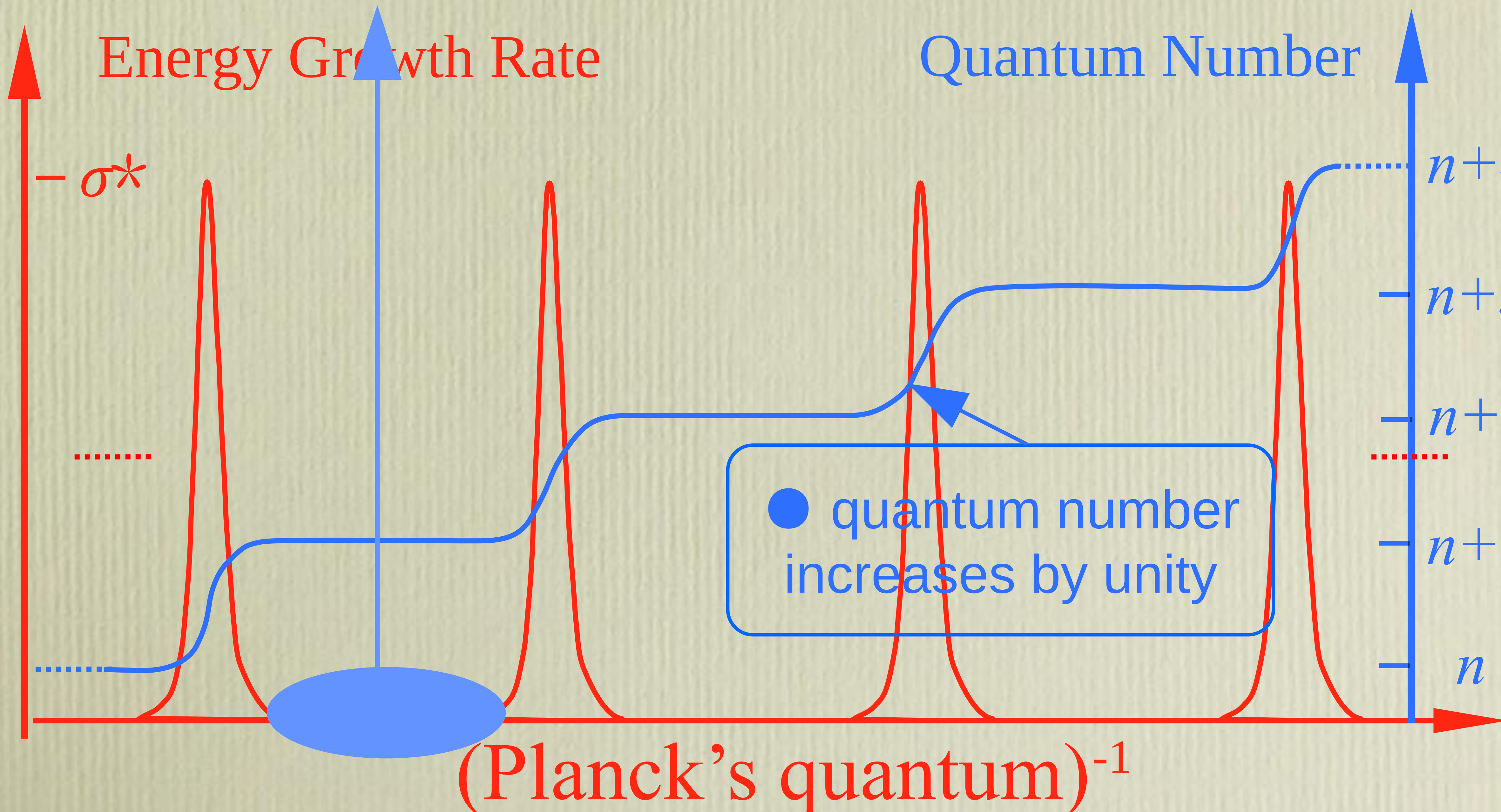
Planck's quantum-driven IQHE(VI)

- insulator characterized by an integer



Planck's quantum-driven IQHE(VII)

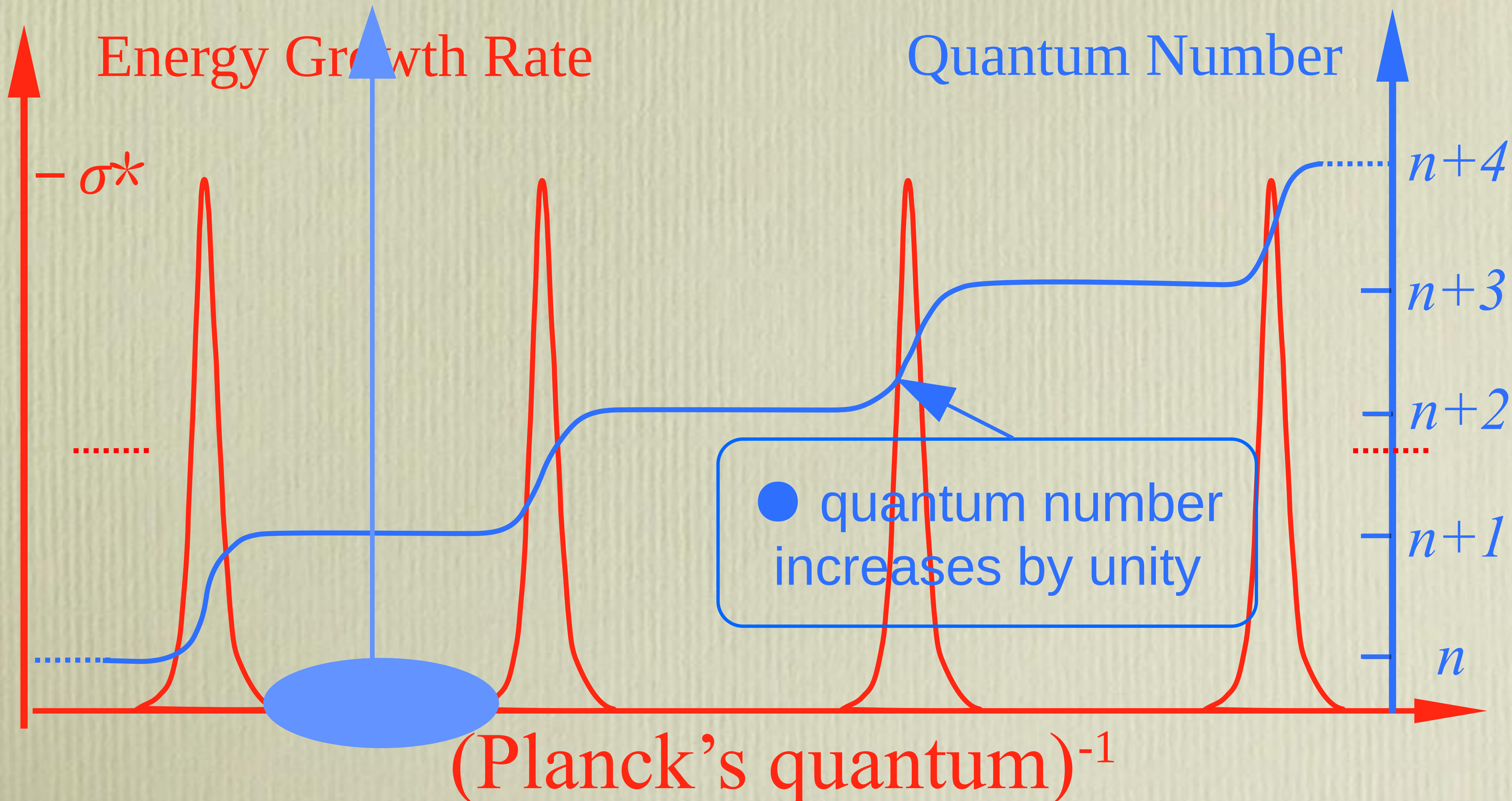
- insulator characterized by an integer



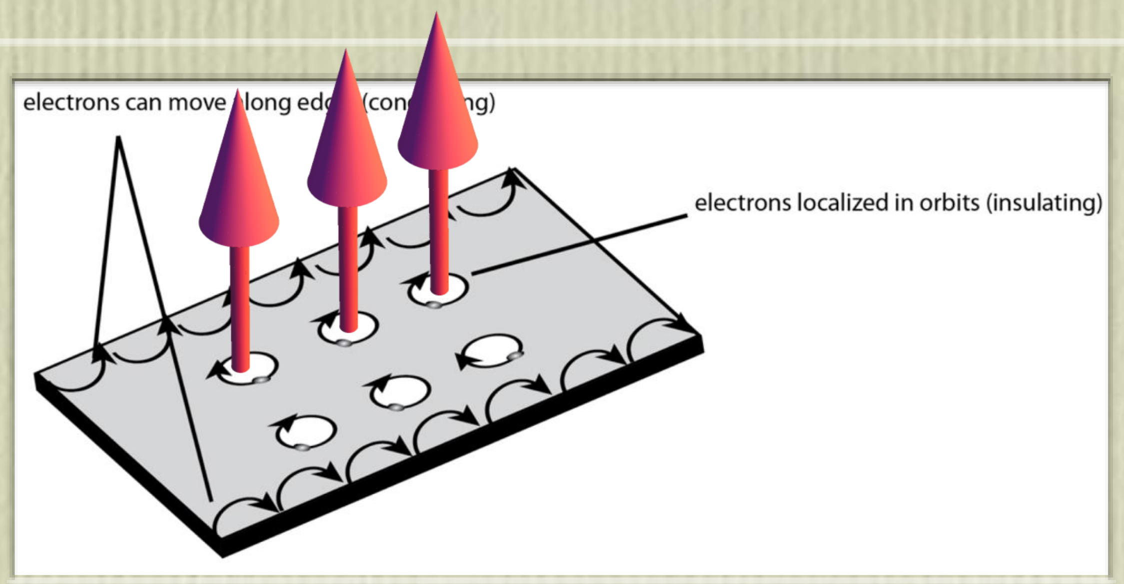
Planck's quantum-driven IQHE(VIII)

- insulator characterized by an integer

- This quantum number is of topological nature.



Integer quantum Hall effect

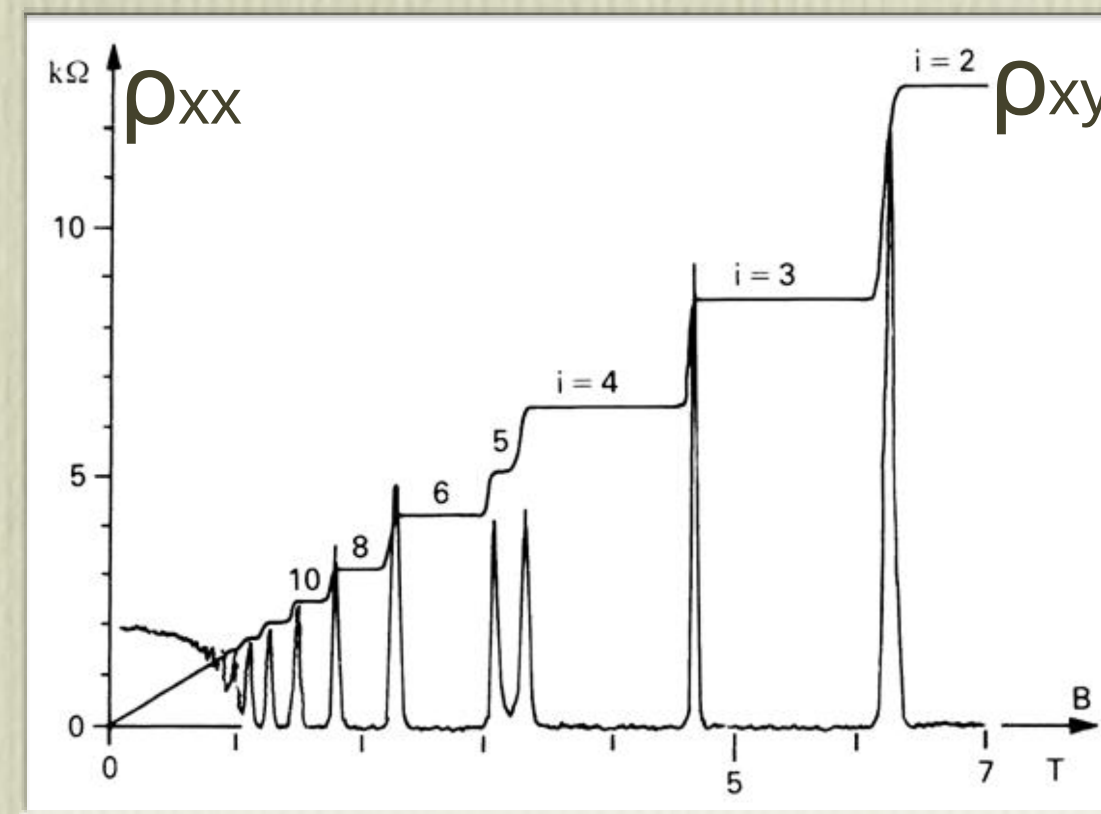


two dimensional electron gas (MOSFET)

strong magnetic field

quantized Hall conductance

Claus von Klitzing



Phenomenological analogy to conventional IQHE

- energy growth rate → longitudinal conductivity
- quantum number → Hall conductivity
- inverse Planck's quantum → filling fraction

Fundamental differences from conventional IQHE

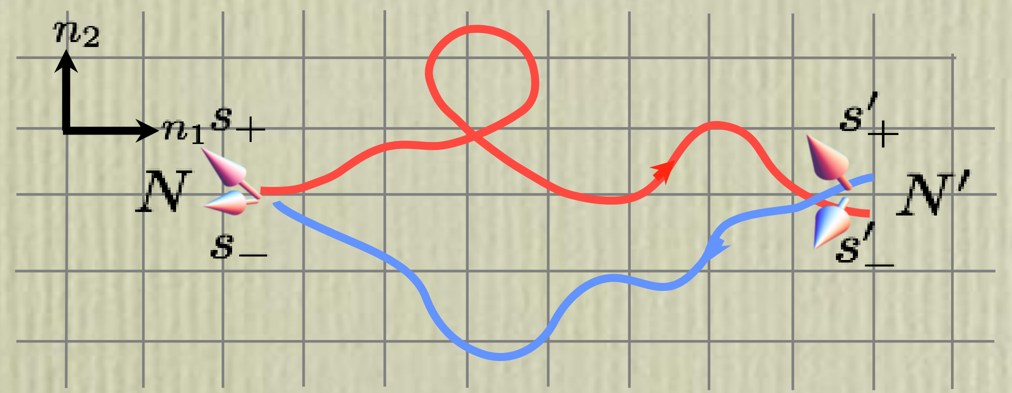
- no magnetic field, no electromagnetic response, driven by Planck’s quantum
- strong chaoticity origin (This phenomenon disappears even when regular quantum dynamics is partially restored.)
- one-body system → no concept such as integer filling
- no translation symmetry, no adiabatic parameter cycle

Analytic theory (I)

- mapping onto 2D periodic quantum dynamics

$$\psi_t = \hat{U}^t \psi_0, \quad \hat{U} \equiv e^{-\frac{i}{h_e}V(\theta_1,\theta_2)} e^{-\frac{i}{2}(h_e \hat{n}_1^2 + 2\bar{\omega}\hat{n}_2)} = e^{-\frac{i}{h_e}V(\hat{\theta}_1,\hat{\theta}_2)} e^{-\frac{i}{h_e}H_0(\hat{n}_1,\hat{n}_2)}$$

$$E(t) = \frac{1}{2}\int \frac{d\omega}{2\pi} e^{-i\omega t} Tr\left(\hat{n}_1^2 K_\omega \psi_0 \otimes \psi_0^+\right)$$



interference between advanced and retarded quantum amplitudes

two-particle Green function

$$K_\omega(Ns_+s'_+; N's_-s'_-) = \langle\langle Ns|G^+(\omega_+)|N's'_+\rangle\langle N's'_-|G^-(\omega_-)|Ns_-\rangle\rangle_{\omega_0}$$

$$G^\pm(\omega_\pm) = (1-(e^{i\omega_\pm}U)^{\pm 1})^{-1} \qquad \omega_\pm = \omega_0 \pm \omega/2$$

Analytic theory (I)

- mapping onto 2D periodic quantum dynamics

$$\psi_t = \hat{U}^t \psi_0, \quad \hat{U} \equiv e^{-\frac{i}{\hbar_e} V(\theta_1,\theta_2)} e^{-\frac{i}{2}(h_e \hat{n}_1^2 + 2\bar{\omega}\hat{n}_2)}$$

$$E(t) = \frac{1}{2} \int \frac{d\omega}{2\pi} e^{-i\omega t} Tr\left(\hat{n}_1^2 K_\omega \psi_0 \otimes \psi_0^+\right)$$

- exact expression for K_ω - functional integral over supermatrix field (color-flavor transformation, Zirnbauer, '96)

$$K_\omega\left(Ns_+ s_-, N' s'_+ s'_-\right)$$
$$= \int D(Z,\tilde{Z}) e^{-S[Z,\tilde{Z}]} \left(\left(1 - Z\tilde{Z}\right)^{-1} Z\right)_{Ns_+ b, Ns_- b} \left(\left(1 - \tilde{Z}Z\right)^{-1} \tilde{Z}\right)_{N's'_+ b, N's'_- b}$$

$$S[Z,\tilde{Z}] = -\mathrm{Str}\ln(1 - Z\tilde{Z}) + \mathrm{Str}\ln(1 - e^{i\omega}\hat{U} Z \hat{U}^\dagger \tilde{Z})$$

Analytic theory (II)

- mapping onto 2D periodic quantum dynamics

$$\psi_t = \hat{U}^t \psi_0, \quad \hat{U} \equiv e^{-\frac{i}{h_e} V(\theta_1, \theta_2)} e^{-\frac{i}{2}(h_e \hat{n}_1^2 + 2\tilde{\omega}\hat{n}_2)}$$

$$E(t) = \frac{1}{2}\int \frac{d\omega}{2\pi} e^{-i\omega t} Tr\left(\hat{n}_1^2 K_\omega \psi_0 \otimes \psi_0^+\right)$$

- chaos (fast correlation decay) → local field Z(N)

Analytic theory (III)

- mapping onto 2D periodic quantum dynamics

$$\psi_t = \hat{U}^t \psi_0, \quad \hat{U} \equiv e^{-\frac{i}{h_e}V(\theta_1,\theta_2)} e^{-\frac{i}{2}(h_e \hat{n}_1^2 + 2\bar{\omega}\hat{n}_2)}$$

$$E(t) = \frac{1}{2}\int \frac{d\omega}{2\pi} e^{-i\omega t} Tr\left(\hat{n}_1^2 K_\omega \psi_0 \otimes \psi_0^+\right)$$

- K_ω - functional integral over Z(N)

$$Q = \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix} \Lambda \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}^{-1}$$

$$K_\omega\left(Ns_+s_-, N's'_+ s'_-\right) = -\frac{1}{4}\delta_{s'_+s'_-}\int D(Q) e^{-S[Q]} Q(N)_{+b,-b} Q(N')_{-b,+b}$$

$$S[Q] = \frac{1}{4}\mathrm{Str}\left(-\sigma(\nabla Q)^2 + \sigma_{\mathrm{H}} Q\nabla_1 Q\nabla_2 Q - 2i\omega Q\Lambda\right)$$

independent of H_0

Analytic theory (III)

- mapping onto 2D periodic quantum dynamics

$$\psi_t = \hat{U}^t \psi_0, \quad \hat{U} \equiv e^{-\frac{i}{\hbar_e} V(\theta_1,\theta_2)} e^{-\frac{i}{2}(h_e \hat{n}_1^2 + 2\bar{\omega}\hat{n}_2)}$$

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topological θ-term

Analytic theory (III)

- mapping onto 2D quantum dynamics

$$\psi_t = \hat{U}^t \psi_0, \quad \hat{U} \equiv e^{-\frac{i}{h_e} V(\theta_1,\theta_2)} e^{-\frac{i}{2}(h_e \hat{n}_1^2 + 2\bar{\omega}\hat{n}_2)}$$

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- K_ω - functional integral over Z(N)

$$K_\omega\left(Ns_+s_-, N's'_+ s'_-\right) = -\frac{1}{4}\delta_{s'_+s'_-}\int D(Q) e^{-S[Q]} Q(N)_{+b,-b} Q(N')_{-b,+b}$$

$$S[Q] = \frac{1}{4}\mathrm{Str}\left(-\sigma(\nabla Q)^2 + \sigma_{\mathrm{H}} Q\nabla_1 Q \nabla_2 Q - 2i\omega Q\Lambda\right)$$

$\sigma_{\mathrm{H}} \propto h_e^{-1}$ "classical Hall conductivity"

Analytic theory (IV)

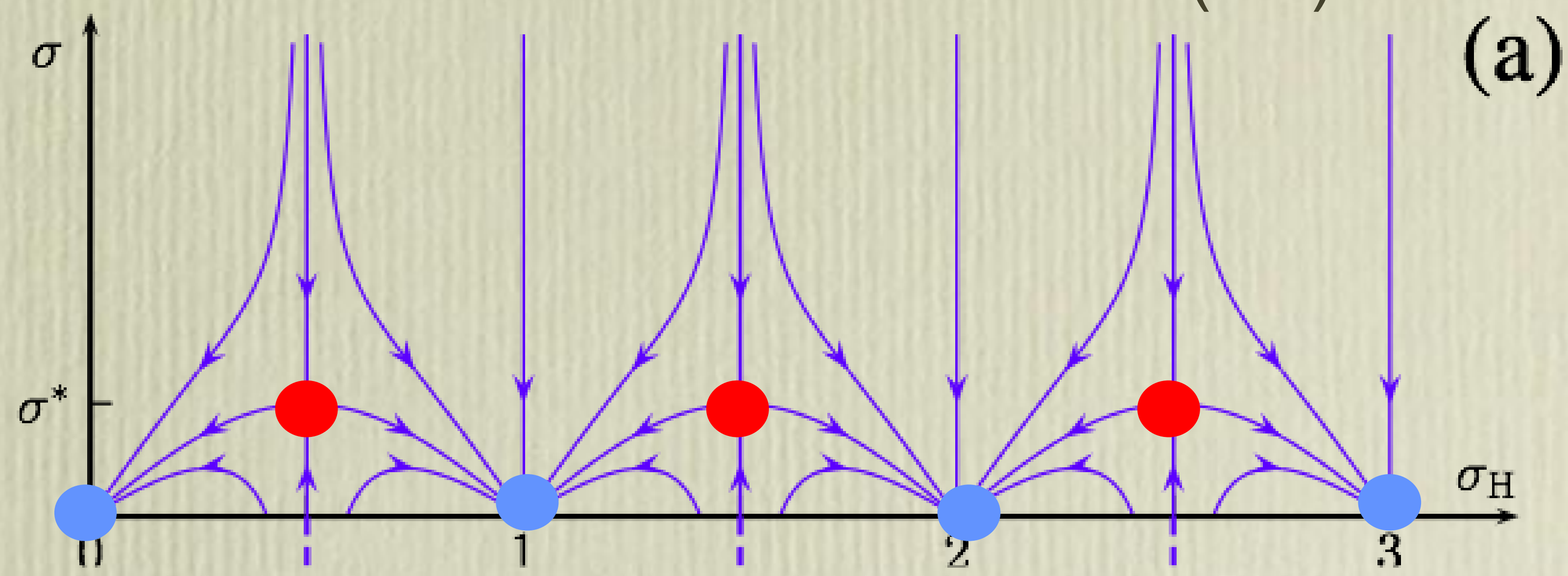
• background field formalism (Pruisken ‘80s)
• instanton method (Burmistrov and Pruisken ‘05)

$$\frac{d\tilde{\sigma}}{d\ln\tilde{\lambda}} = -\frac{1}{8\pi^2\tilde{\sigma}} - \frac{32\pi}{e}\tilde{\sigma}^2 e^{-4\pi\tilde{\sigma}} \cos 2\pi\tilde{\sigma}_{\mathrm{H}}$$

$$\frac{d\tilde{\sigma}_{\mathrm{H}}}{d\ln\tilde{\lambda}} = -\frac{64\pi}{e}\tilde{\sigma}^2 e^{-4\pi\tilde{\sigma}} \sin 2\pi\tilde{\sigma}_{\mathrm{H}}$$

$$\sigma^* \approx 0.44$$

Khmelnitskii’s RG flow (‘83)



● insulating phase: $\sigma=0$, $\sigma_H=n$ (emergent quantum number)

● metallic phase: $\sigma=\sigma^*$, $\sigma_H=n+1/2$

$$\sigma_{\mathrm{H}} \propto h_e^{-1}$$

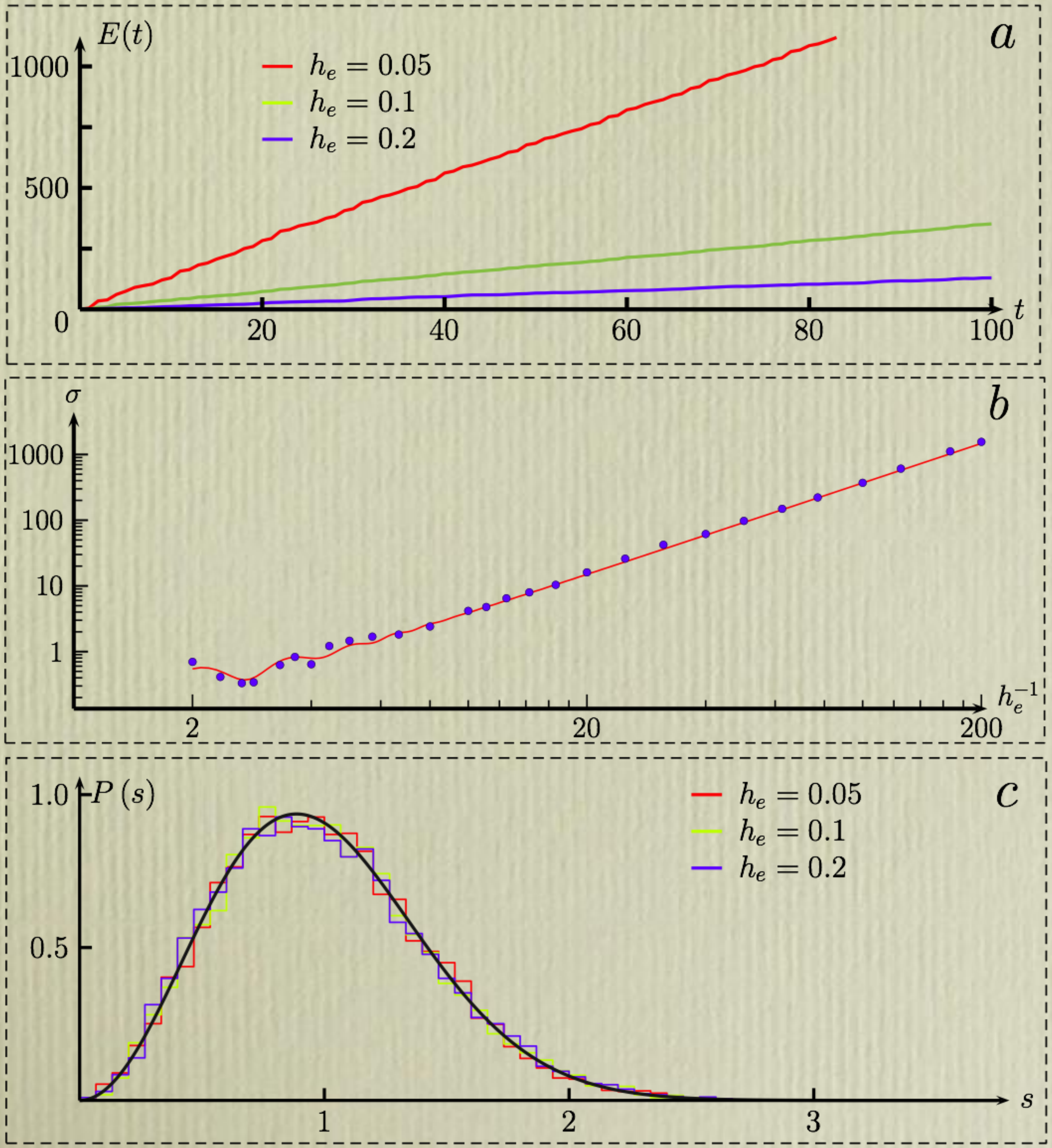
“classical Hall conductivity” ⟶ staircase-like pattern

Numerical test ($t<10^2$): chaoticity

$$V = \frac{2\arctan 2d}{d} d\cdot\sigma,$$

$$d = \left(\sin\theta_1, \sin\theta_2, 0.8(1-\cos\theta_1-\cos\theta_2)\right)$$

Beenakker et. al. '11

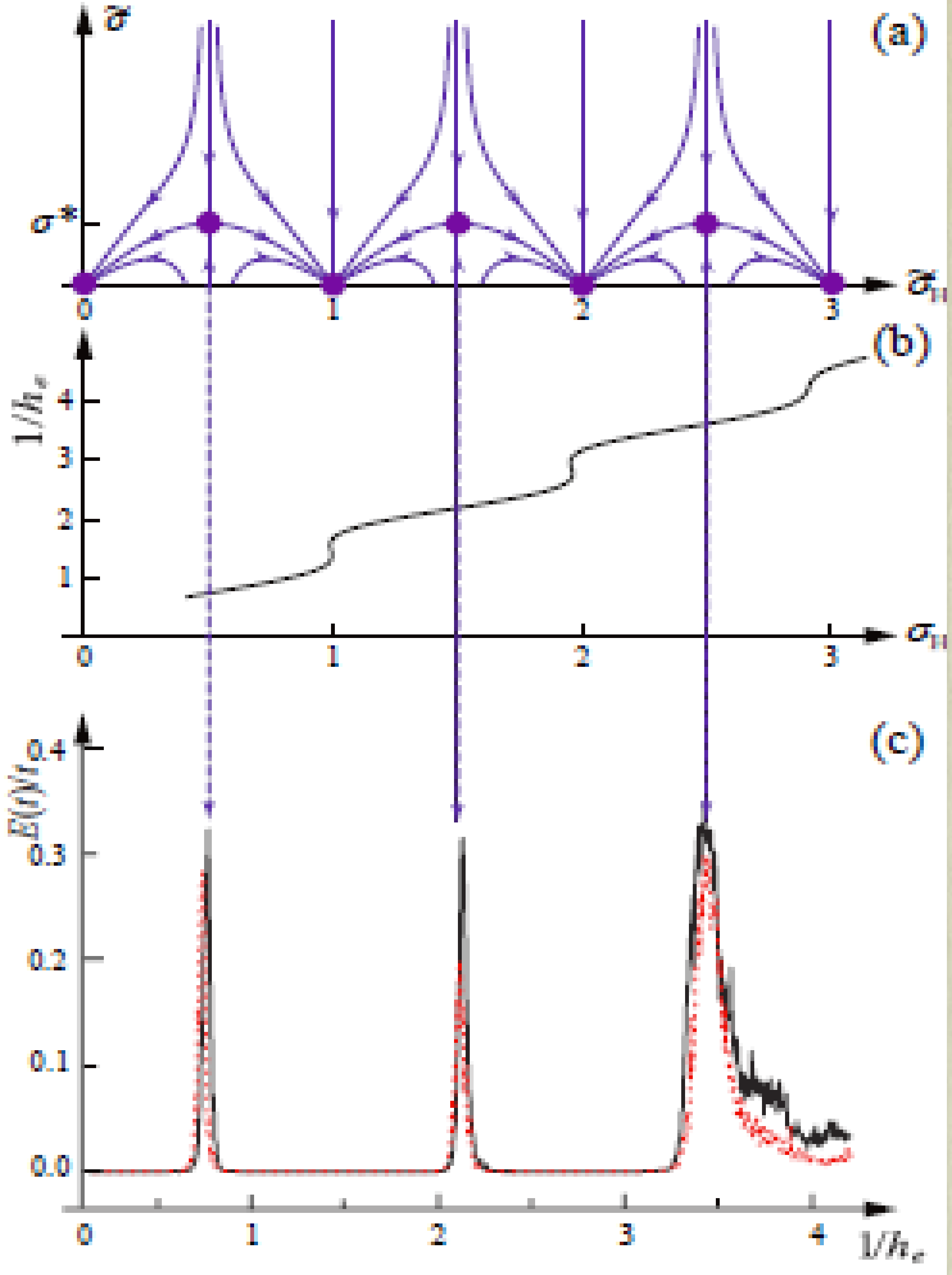


☛ linear energy growth in short times

☛ blue dots are simulation results for the energy growth rate in short times;

☛ red line is the theoretical prediction.

☛ fluctuations of eigen quasi-energies follow Wigner-Dyson statistics of unitary type.



Numerical test ($t<6\times10^7$): transition between topological insulating phases

Hall plateaux (n=0,1,2,…)

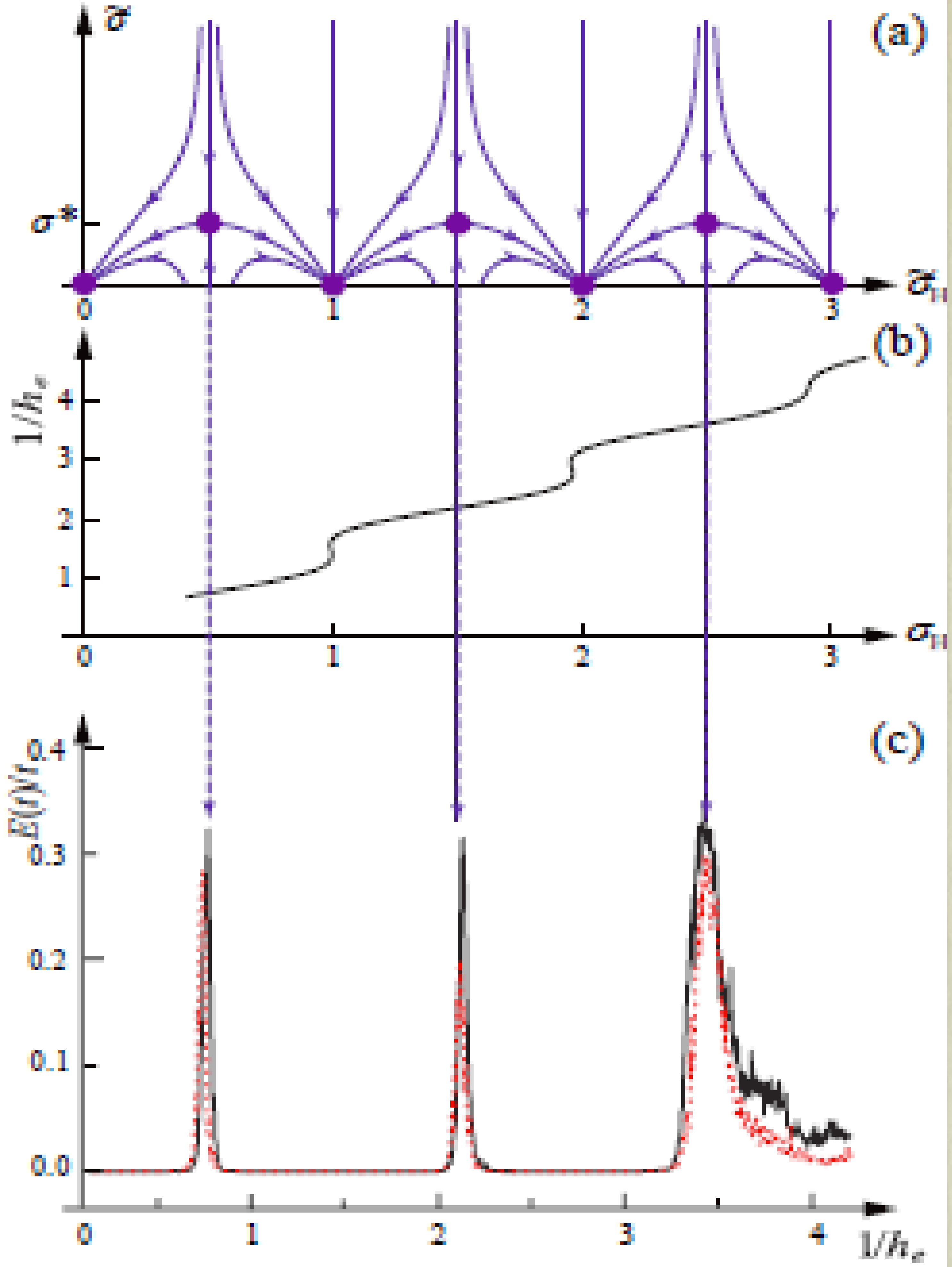
critical points (n=1/2,3/2, …)

☛ Analytic results for $\sigma_H(h_e)$ predict three transition points at $1/h_e=0.73,2.19,3.60$ for $0.23< h_e <1.50$.

☛ Simulations indeed show three transition points at $1/h_e =0.77,2.13,3.45$.

☛ Simulations show that the growth rate at the critical point is universal.

(solid line)	$H_0=(h_e n_1)^2$
(dotted line)	$H_0=(h_e n_1)^4$



Numerical test ($t<6\times10^5$): transition between topological insulating phases

Hall plateaux (n=0,1,2,…)

critical points (n=1/2,3/2, …)

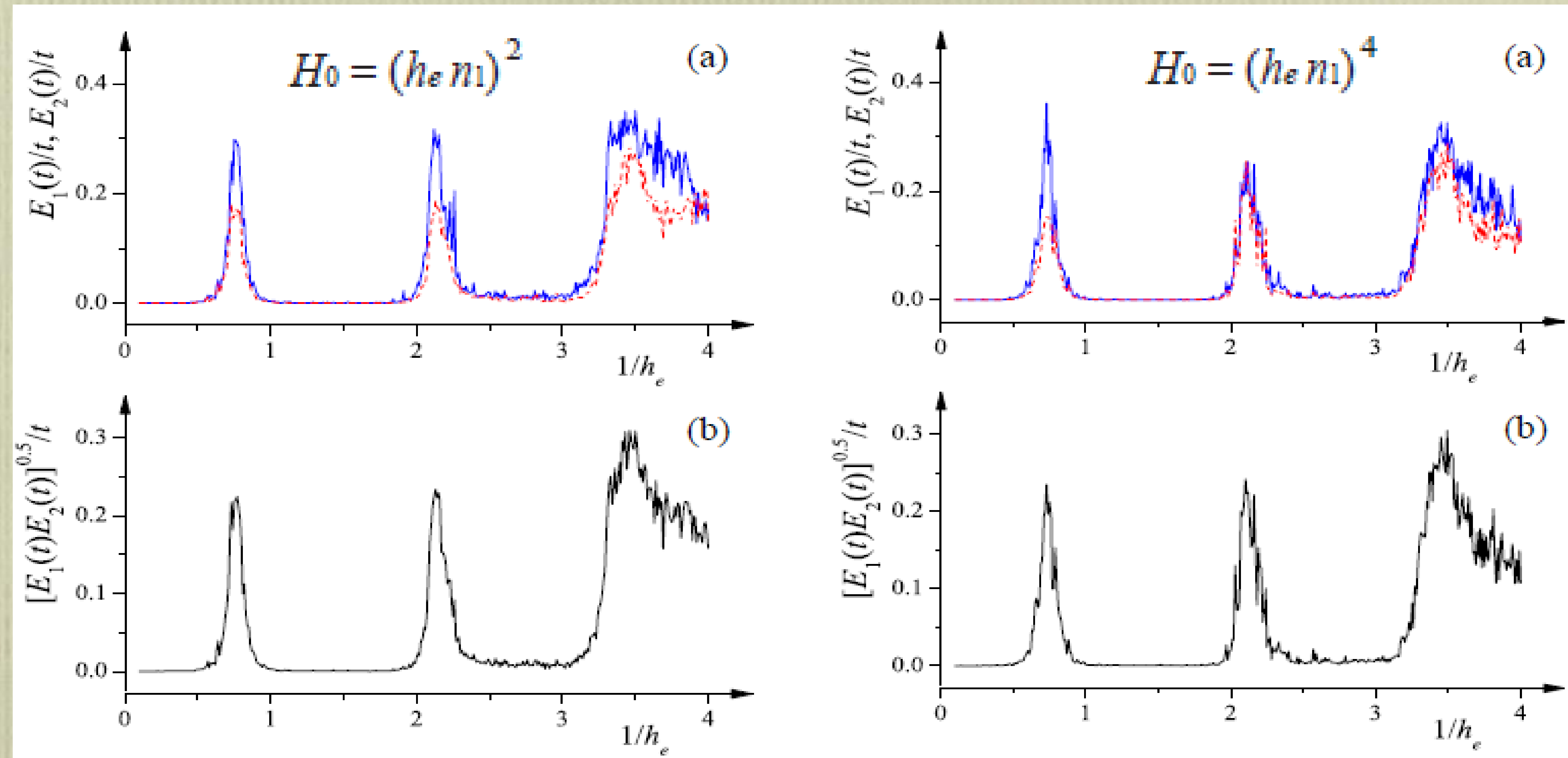
☛ Analytic results for $\sigma_H(h_e)$ predict three transition points at $1/h_e=0.73,2.19,3.60$ for $0.23< h_e <1.50$.

☛ Simulations indeed show three transition points at $1/h_e =0.77,2.13,3.45$.

☛ Simulations show that the growth rate at the critical point is universal.

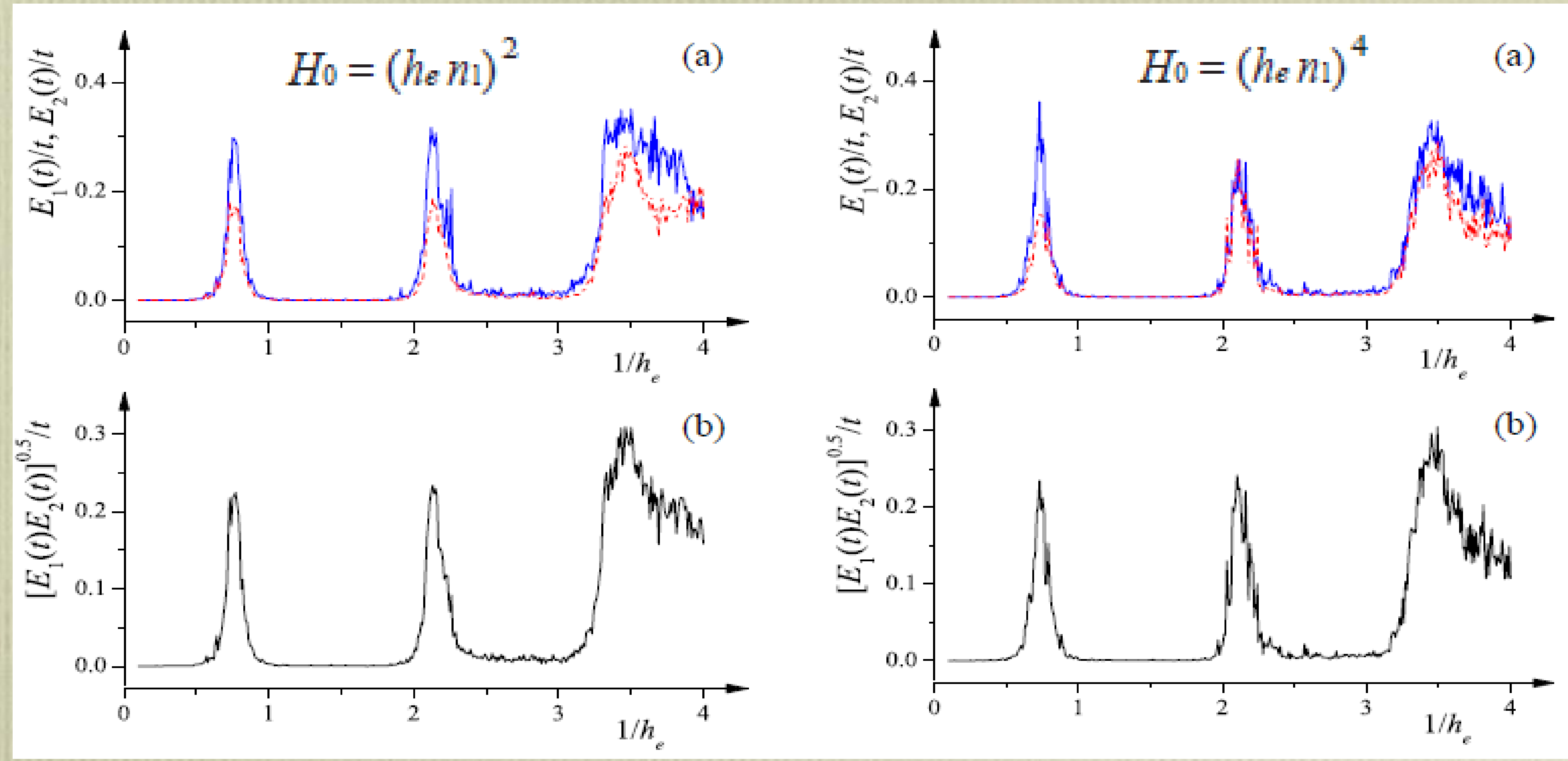
☛ Simulations show that the transition is robust against the change of H_0.

Universality of critical energy growth rate



$H_0 = (-ih_e\partial_{\theta_1})^\alpha$	1st peak	2nd peak	3rd peak
$\alpha = 2$	0.22	0.23	0.30
$\alpha = 4$	0.23	0.24	0.30

Universality of critical energy growth rate



expectation from Chern-Simons theory (Lee, Kivelson, and Zhang '92) of conventional IQHE:

$$\sigma^* = 0.25$$

$$V = \frac{2\arctan 2d}{d} d \cdot \sigma,$$

$$d = \left(\sin\theta_1, \sin\theta_2, 0.8\left(1 - \cos\theta_1 - \cos\theta_2\right)\right)$$

change the value of μ

other conditions not changed

$$V = \frac{2\arctan 2d}{d} d \cdot \sigma,$$

$$d = \left(\sin\theta_1, \sin\theta_2, 0.8\left(1 - \cos\theta_1 - \cos\theta_2\right)\right)$$

change the value of μ

other conditions not changed

$$I = -\frac{1}{4\pi}\iint d\theta_1 d\theta_2 \left(\partial_{\theta_1}\frac{\vec{V}}{|V|} \times \partial_{\theta_2}\frac{\vec{V}}{|V|}\right) \cdot \frac{\vec{V}}{|V|}$$

$$|\mu| > 2 : I = 0;$$

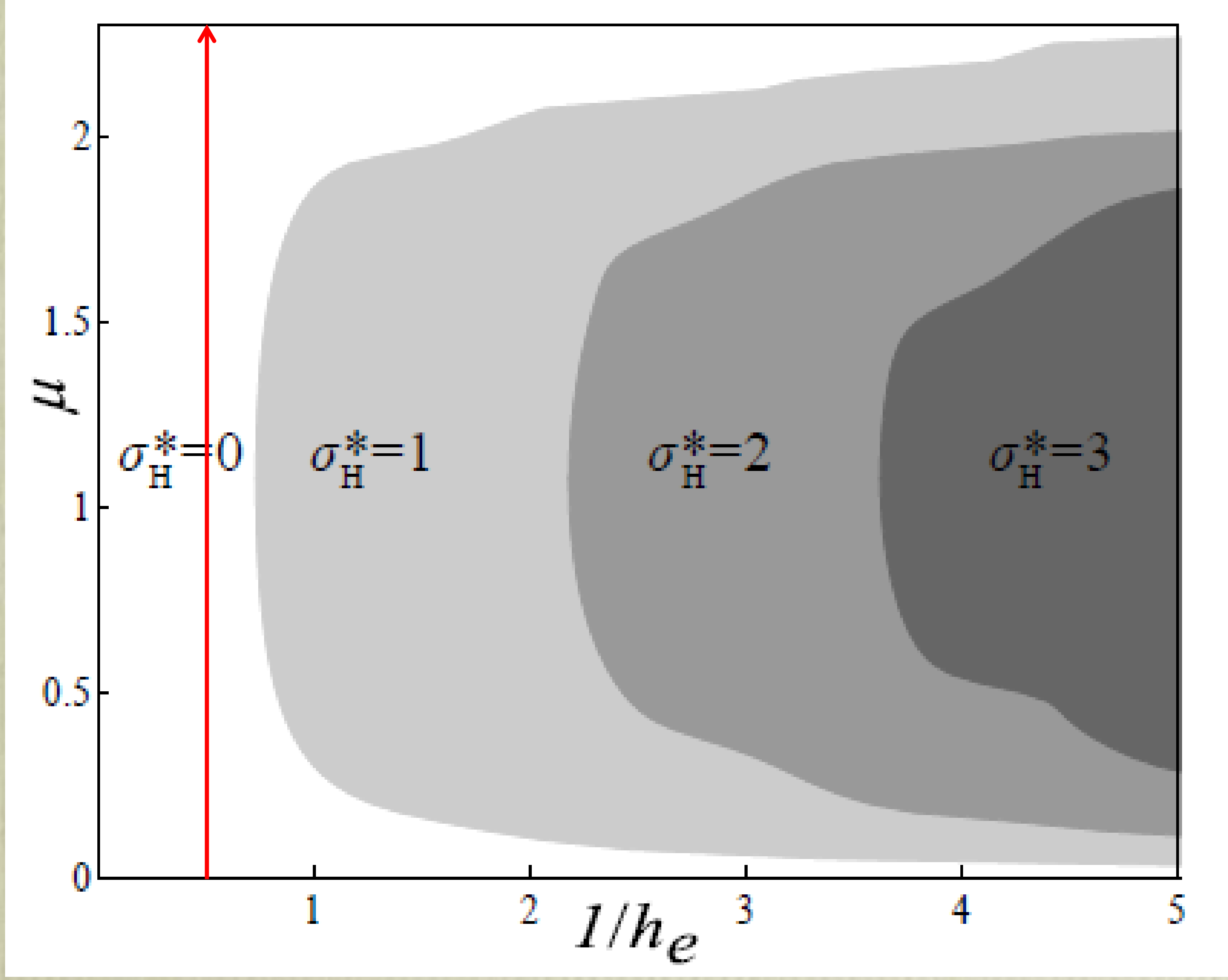
$$|\mu| < 2 : I = +1.$$

Naïve Chern index predicts only two phases, no matter the value of h_e (Beenakker et. al. ‘11).

$$V = \frac{2\arctan 2d}{d} d \cdot \sigma,$$

$$d = \left(\sin\theta_1, \sin\theta_2, 0.8(1 - \cos\theta_1 - \cos\theta_2)\right)$$

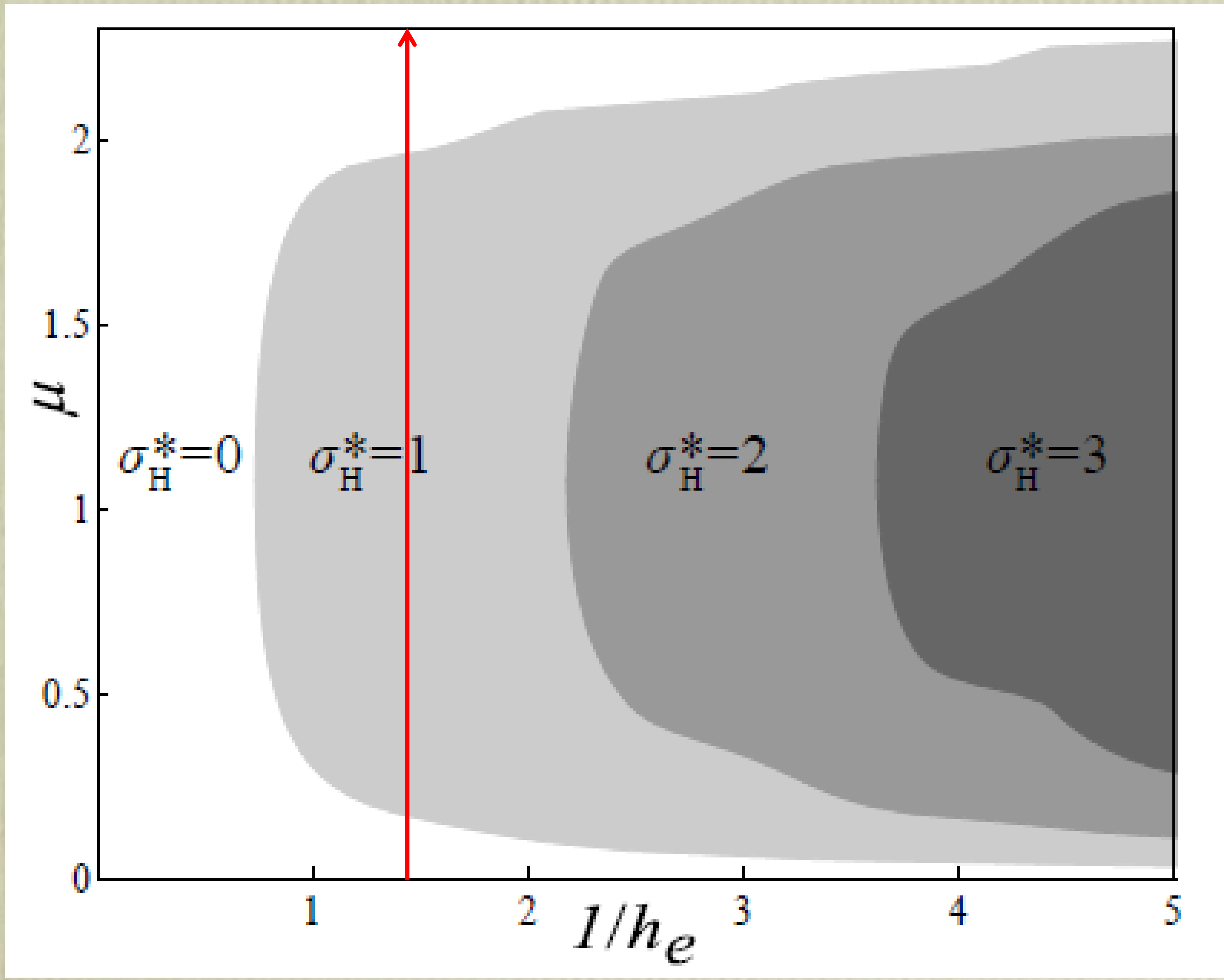
change the value of μ



$$V = \frac{2\arctan 2d}{d} d \cdot \sigma,$$

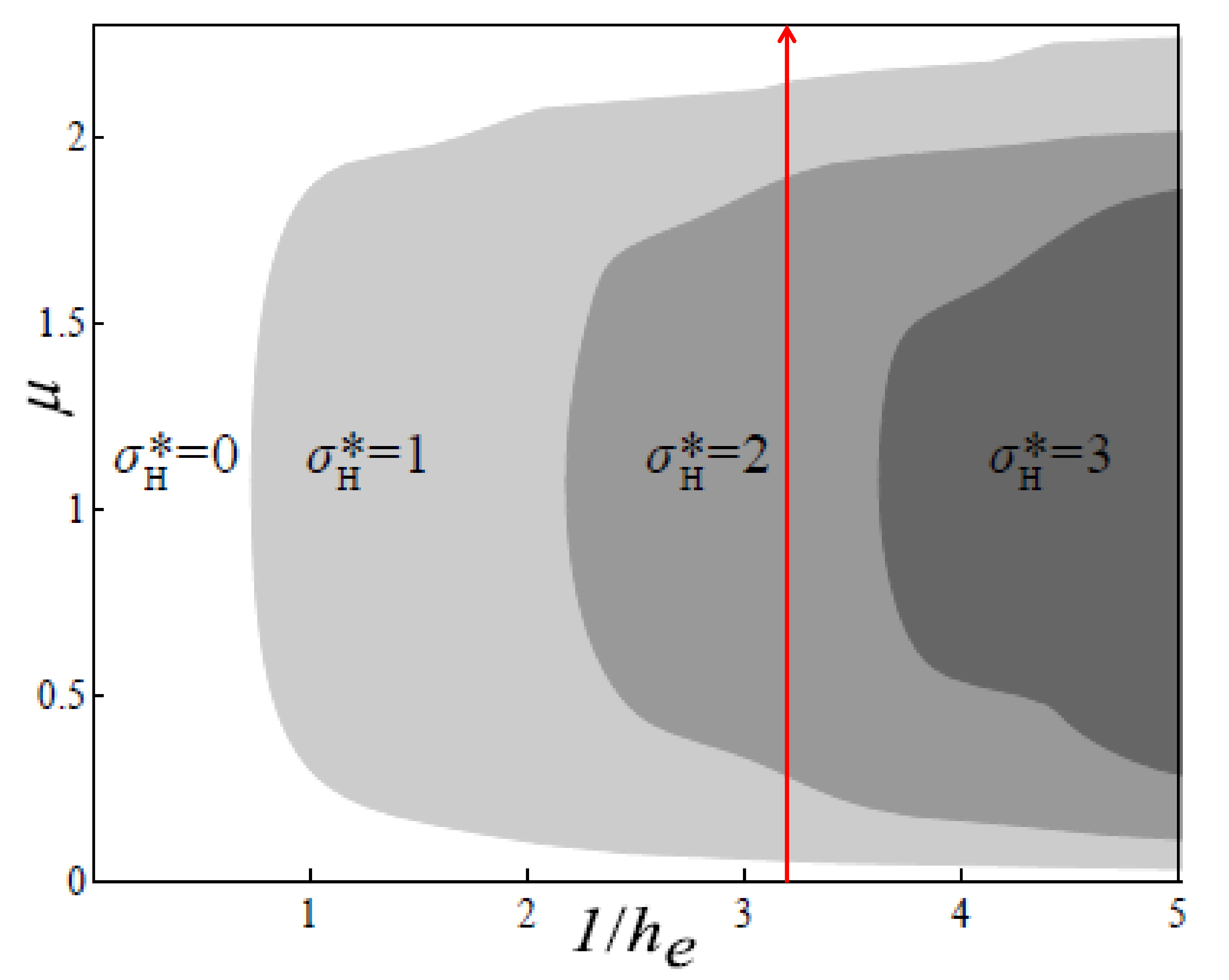
$$d = \left(\sin\theta_1, \sin\theta_2, 0.8\left(1 - \cos\theta_1 - \cos\theta_2\right)\right)$$

change the value of μ



$$V = \frac{2\arctan 2d}{d} d \cdot \sigma,$$
$$d = (\sin\theta_1, \sin\theta_2, 0.8(1 - \cos\theta_1 - \cos\theta_2))$$

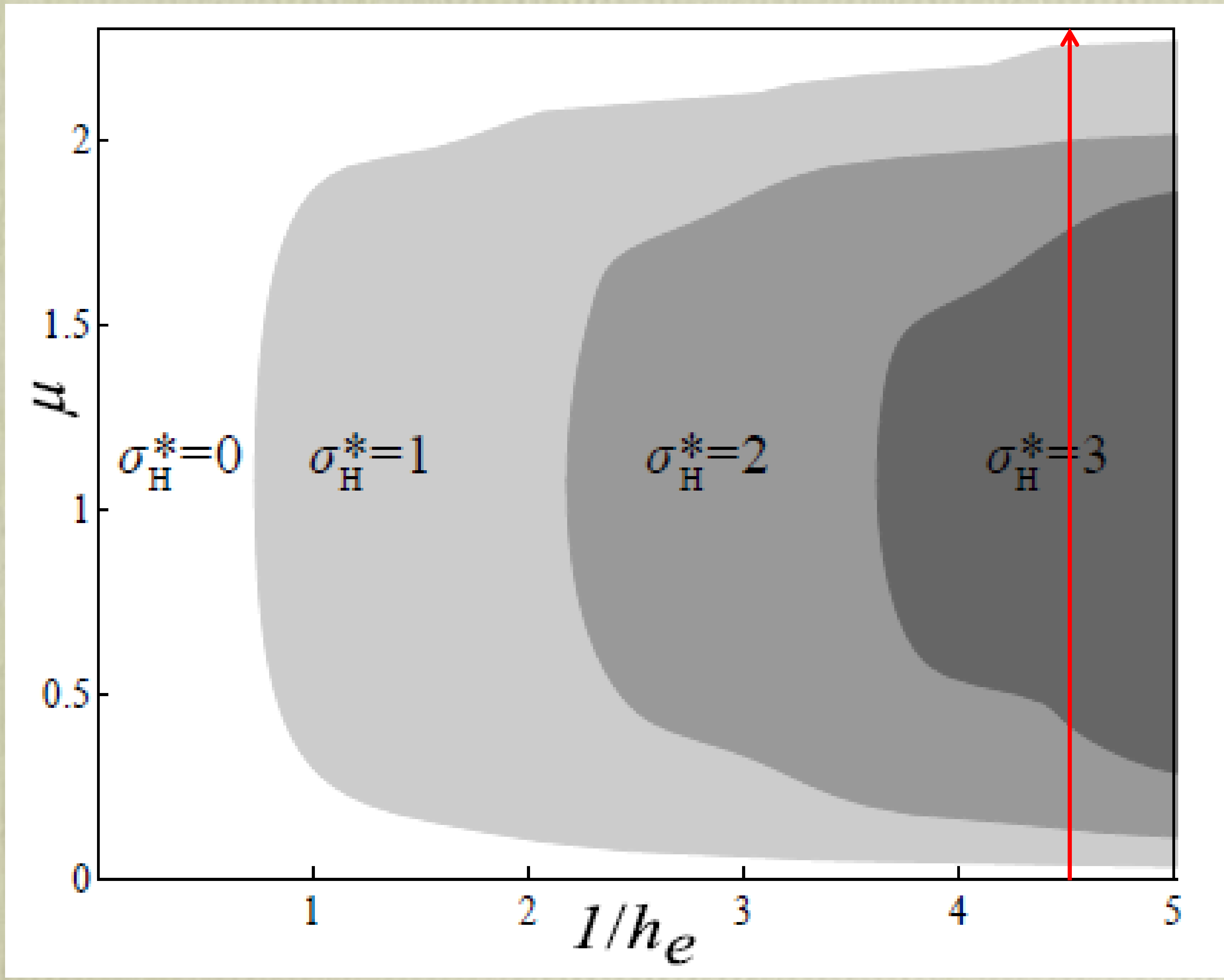
change the value of μ



$$V = \frac{2\arctan 2d}{d} d \cdot \sigma,$$

$$d = (\sin\theta_1, \sin\theta_2, 0.8(1 - \cos\theta_1 - \cos\theta_2))$$

change the value of μ

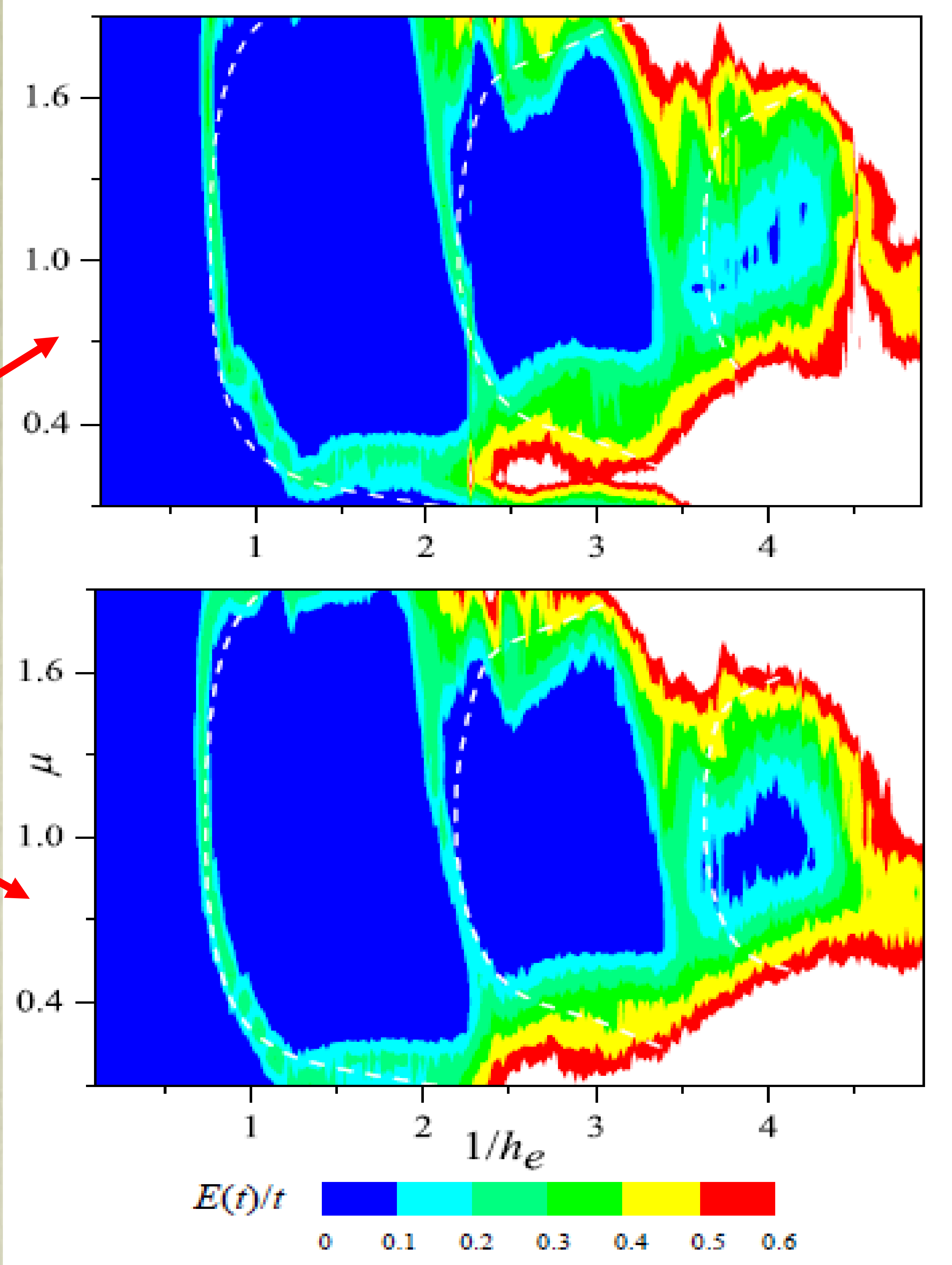


$$V = \frac{2\arctan 2d}{d} d \cdot \sigma,$$

$$d = (\sin\theta_1, \sin\theta_2, 0.8(\mu - \cos\theta_1 - \cos\theta_2))$$

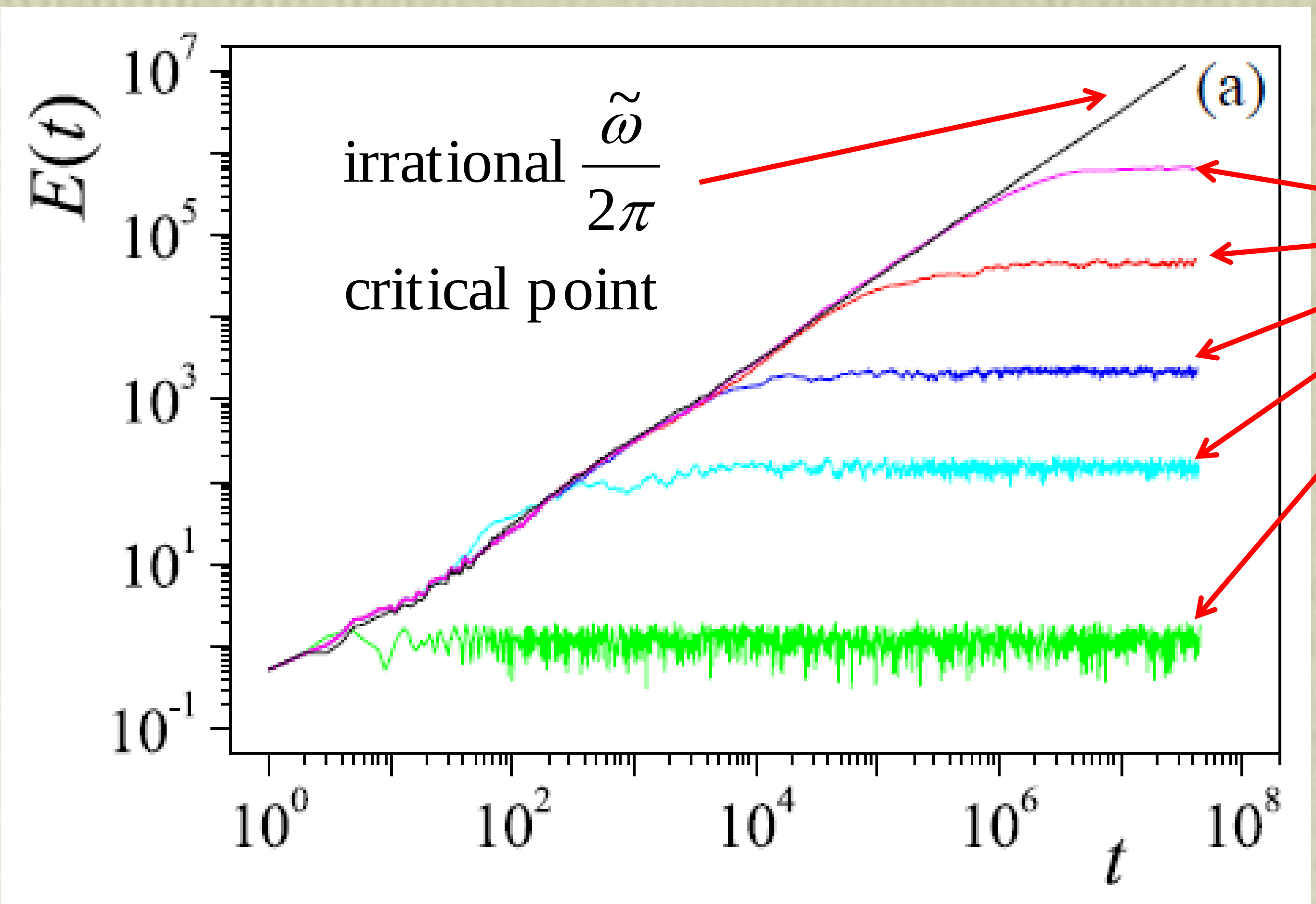
$$H_0 = (h_e n_1)^2$$

$$H_0 = (h_e n_1)^4$$



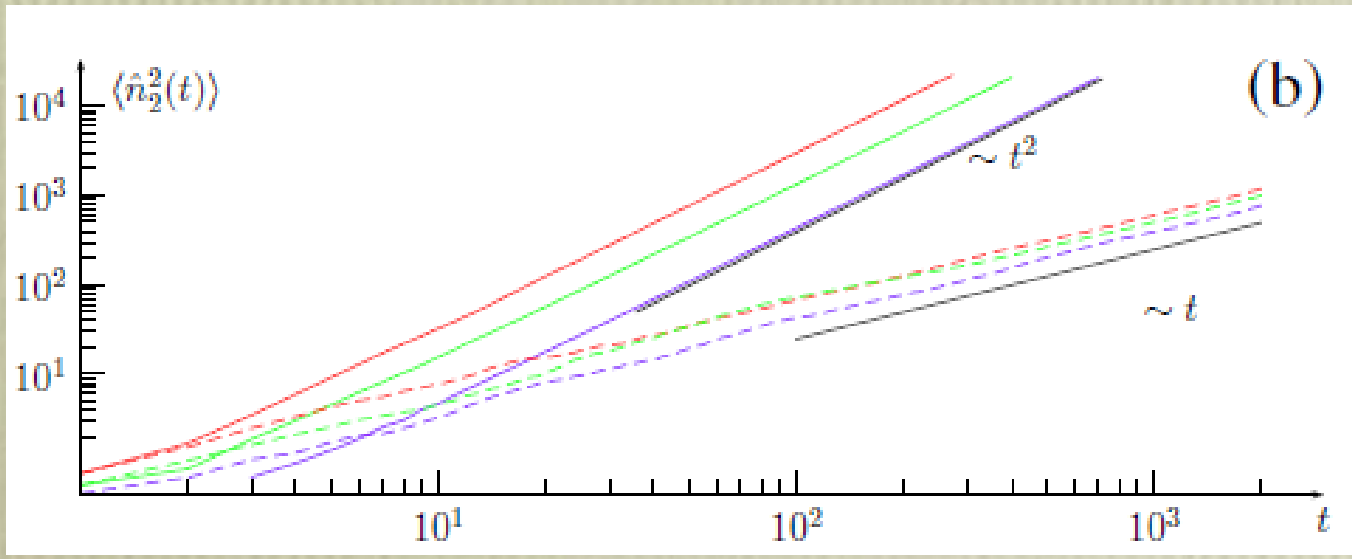
Rich topological phases are excited by chaos.

Destroying fully developed chaos: absence of Planck's quantum-driven IQHE



rational $\frac{\tilde{\omega}}{2\pi} = \frac{p}{q}$

☛ no localization-delocalization transitions occurs; the system is always insulating.



☛ the equivalent 2D system exhibits ballistic motion in the virtual (n_2) direction.

Conclusion

- a deep connection between chaos and IQHE uncovered

- Planck's quantum ↔ magnetic field;

  energy growth rate ↔ longitudinal conductivity;

  hidden quantum number ↔ quantized Hall conductivity;

- strong chaoticity origin

- rich topological quantum phenomena emerge from chaos

Open questions

- the nature of the analogy between the novel transition and connventional IQHE
- nature of universal quantum diffusion
- experimental tests (spin magnetic resonance and cold atomic gases)
- …