Emergent $\mathrm{SL}(2,\mathbb{R})$ symmetry in a system of fermions

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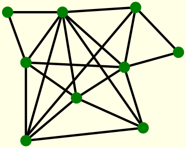
The SYK model



N Majorana operators χ_j

antisymmetric

$$\boxed{\begin{gathered} H = \frac{1}{4!}\sum_{j,k,l,m} J_{jklm}\,\chi_j\,\chi_k\,\chi_l\,\chi_m \\[2ex] \overline{J_{jklm}} = 0, \qquad \overline{J_{jklm}^2} = 3!\,\frac{J^2}{N^3} \end{gathered}}$$

- Sachdev, Ye, 1993 – a similar model with $\mathrm{SU}(M)$spins and two-body interactions.
  - representation of $\mathrm{SU}(M)$ by complex femions
- This model (Kitaev, 2015):
  - The same Green function $G(\tau) = -\langle \mathbf{T}\,\chi_j(\tau)\chi_j(0)\rangle$
  - disorder effects (replica-off-diagonal terms) are negligible
- Detailed calculations: Maldacena, Stanford, `arxiv:1604.07818`

Why is this model interesting?

- Nontrivial but solvable:
  - Dynamic mean field approximation for large N
  - Complete analytic solution for $N \gg \beta J \gg 1$

quantum fluctuations
are small

Analytic solution of
the DMF equations

  - Strongly correlated but not glassy

- Emergent conformal symmetry for $\beta J \gg 1$:
  - Equations: $\mathrm{Diff}(S^1)$
  - Solutions: $\mathrm{PSL}(2,\mathbb{R})$

- The same universality class as black holes

A hint toward the conformal symmetry

- It is convenient to consider q-body interactions:

$$H = \frac{1}{q!} \sum_{j_1,\dots,j_q} J_{j_1\dots j_q}\, \chi_{j_1} \cdots \chi_{j_q}, \qquad \overline{J^2_{j_1\dots j_q}} = (q-1)! \frac{J^2}{N^{q-1}}$$

$$\boxed{\Delta = \frac{1}{q}}$$

- Green function:
  - "Zero-temperature" case, i.e. $J^{-1} \ll \tau \ll \beta$ (Sachdev, Ye, 1993):

$$G_\infty(\tau) = -b^\Delta (J\tau)^{-2\Delta} \quad \text{where } b = \tfrac{1}{\pi}\left(\tfrac{1}{2} - \Delta\right) \tan(\pi\Delta)$$

  - For $\tau \sim \beta \gg J^{-1}$, the Green function is analogous to CFT correlators (Parcollet, Georges, 1998):

$$G(\tau) = -b^\Delta \left(\frac{\beta J}{\pi} \sin \frac{\pi\tau}{\beta} \right)^{-2\Delta} \qquad (\text{for } 0 < \tau < \beta).$$

Thermodynamic properties

- Low-temperature expansion (in powers of $(\beta J)^{-1}$):

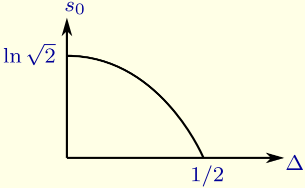
$$\frac{\beta F}{N} = -\frac{\ln Z}{N} = E_0\,\beta J - s_0 - \underbrace{\frac{\gamma}{2}(\beta J)^{-1}}_{} + \cdots$$

energy at $T = 0$

"zero-temperature" entropy

specific heat is proportional to T

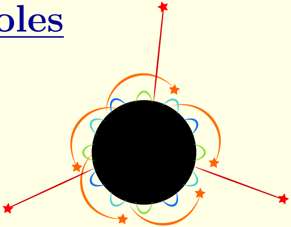
- $s_0 = \pi \int_{\Delta}^{1/2} \left(\frac{1}{2} - x\right) \tan(\pi x)\, dx$



(Parcollet, Georges, Sachdev, 2000 for the original spin model)

- Higher-order terms include fractional poweres of $(\beta J)^{-1}$

Connection to black holes



- Hawking radiation: The black hole horizon is a special type of heat bath

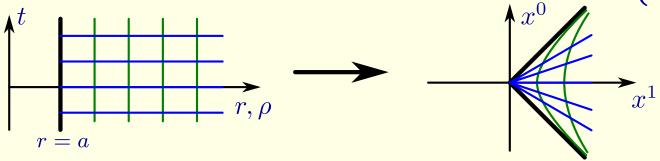
Outline of the derivation:

$$d\ell^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \quad \text{(where } f(r) = 1 - \frac{a}{r} \text{ in } 3+1 \text{ D)}$$

  – Metric is smooth near the horizon:

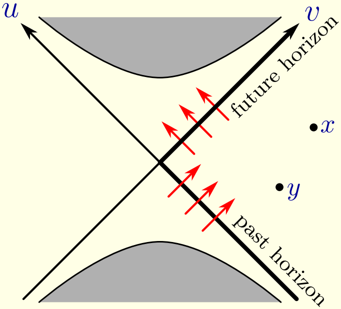
$$d\ell^2 \approx \underbrace{-\varkappa^2\rho^2 dt^2 + d\rho^2}_{-(dx^0)^2 + (dx^1)^2,} + a^2 d\Omega^2$$

$$\begin{cases} x^0 = \rho\sinh(\varkappa t), \\ x^1 = \rho\cosh(\varkappa t) \end{cases}$$



Surface gravity: $\varkappa$ = (time at $r = \infty$)/(Lorentz boost near $r = a$)

Hawking radiation (cont.)



- Causal correlators of free bosons or fermions, e.g. $\langle[\psi_\alpha(x), \psi_\beta(y)]\rangle$, are found by solving the wave equation.

- One can also find $\langle\psi_\alpha(x)\psi_\beta(y)\rangle$ from the quantum state on the past horizon:

– Assume that the correlators on the past horizon are just like on the light cone in flat space-time:

$$\langle\psi_\to(u,0)\,\psi_\to(u',0)\rangle \sim (u-u')^{-1} \quad \text{(for fermions in } 1+1 \text{ D)}$$

– If $u = -e^{-\varkappa t}$, then $\langle\psi_\to(t)\,\psi_\to(t')\rangle \sim \left(\sin\frac{\pi(t-t')}{\beta}\right)^{-1}$, $\boxed{\beta = \frac{2\pi}{\varkappa}}$

Black hole information paradox

- From classical gravity and thermodynamics,

$$T = 2\pi\varkappa \quad \Rightarrow \quad S = \frac{A}{4}$$

area in
Plank units

- Suppose that the black hole forms from a system of particles in a pure quantum state and evaporates completely.
  - According to Hawking's theory, the radiation is an a mixed state with entropy S.
  - By unitarity, the radiation should be in some (very complex) pure state.

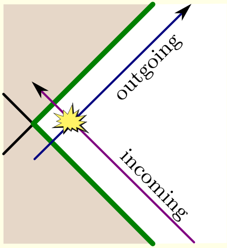
- The same discrepancy exists between a Gaussian heat bath model and the exact quantum dynamics of, say, a chunk of metal. However, in the black hole case, there is no obvious way for the input state to influence the radiation.

- A complete solution will likely require a full quantum theory of gravity.

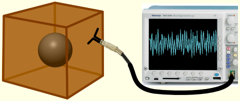
A partial (semicalssical) solution to the paradox

- Gravitational interaction between incoming matter (and fall-back radiation) with the outgoing radiation. Idea by Drey and t'Hoooft (1985), t'Hooft (1986).

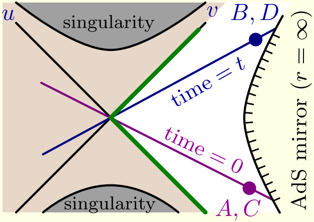
  Amplified by Lorentz factor: $\gamma = e^{\varkappa t}$



- Some well-defined questions were formulated and answered by Shenker and Stanford (2013, 2015).



- However, t'Hooft's effect does not alter the quantum state on the past horizon or any "physical" correlators. It shows in *out-of-time-ordered* (OTO) correlators like $\langle D(t)C(0)B(t)A(0)\rangle$.



Keldysh vs. OTO correlators

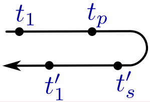
- Keldysh correlators arise in a very natural setting:

$$H = H_{\text{system}} + H_{\text{probe}} - V, \qquad V = \sum_j \underset{\substack{\nearrow\\ \text{system}}}{X_j}\,\underset{\substack{\nwarrow\\ \text{probe}}}{Y_j}$$

Expectation values $\operatorname{Tr}(U^\dagger \Pi U \rho_0)$, $\quad U = \mathbf{T}\exp\left(-i\int V(t)\,dt\right)$ expand into terms like this:

$$\left\langle X_{j_1}(t'_1)\cdots X_{j_s}(t'_s)\,X_{k_p}(t_p)\cdots X_{k_1}(t_1)\right\rangle_{\text{system}}$$

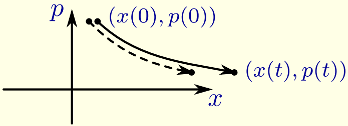
$$t'_1 < \cdots < t'_s, \quad t_p > \cdots > t_1$$



- OTO correlators (for a single electron in the semiclassical regime) were discussed by Larkin and Ovchinnikov (1969).

They characterize the divergence of classical trajectories:

$$[p_j(t), p_k(0)] = i\hbar\,\frac{\partial p_j(t)}{\partial x_k(0)} \sim \boxed{\hbar\, e^{\varkappa t}}$$

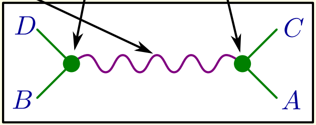


OTO correlators in many-body systems

- For typical non-integrable systems with all-to-all interactions:
  - Early times (but after the two-point correlators have decayed):

$$\langle D(t)C(0)B(t)A(0)\rangle \approx \langle DB\rangle\langle CA\rangle + \underbrace{a\,e^{\varkappa t}}\;\underbrace{v_-(D,B)}\;\underbrace{v_+(C,A)}$$

  - The growth saturates when the second term becomes as large as the first.



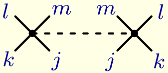
- For black holes and the SYK model:
  - The exponent $\boxed{\varkappa = \frac{2\pi}{\beta}}$ saturates the general bound, $\varkappa \leqslant 2\pi/\beta$ (due to Shenker, Standord, and Maldacena, 2015).
  - The coefficient $\boxed{a \sim i\,\frac{\beta J}{N}}$ is purely imaginary.
  - Special form of the vertex functions: $v_\pm(Y,X) = \langle [P_\pm, Y]\,X\rangle$, where $i[H,P_\pm] = \pm\frac{2\pi}{\beta}P_\pm$.

Diagrams for the SYK model

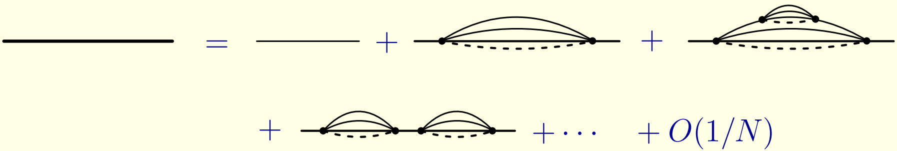
- High-temperature expansion (generally applicable if $\beta J \ll 1$, but also works for $\beta J \ll N$ by suitable resummation)

$$S_{\mathrm{E}}[\chi] = \int_0^\beta \Bigl(\underbrace{\frac{1}{2} \sum_j \chi_j \, \partial_\tau \chi_j}_{\text{zeroth approximation}} + H \Bigr) d\tau$$

$$G_0^{-1} = -\partial_\tau \quad \text{(i.e.} \quad G_0^{-1}(\tau_1,\tau_2) = -\delta'(\tau_1 - \tau_2))$$



$$\overline{J_{jklm}^2} = 3!\,\frac{J^2}{N^3}$$

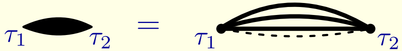


$$= \quad + \quad + \quad + \quad + \quad + \cdots \quad + O(1/N)$$

Resummation of leading diagrams

- Schwinger-Dyson equations:

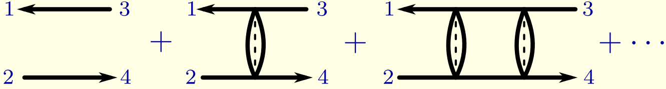
$$G^{-1} = \underbrace{G_0^{-1}}_{\substack{\text{may be neglected}\\ \text{if } \beta J \gg 1}} - \Sigma, \qquad \boxed{-\Sigma(\tau_1,\tau_2) = J^2\big(-G(\tau_1,\tau_2)\big)^{q-1}} \quad \text{(for } q = 3\text{)}$$



- The leading terms in the connected 4-point function are proportional to $1/N$:

$$N^{-2}\sum_{j,k}\big\langle \mathbf{T}\,\chi_j(\tau_1)\,\chi_j(\tau_2)\,\chi_k(\tau_3)\,\chi_k(\tau_4)\big\rangle$$
$$= G(\tau_1,\tau_2)\,G(\tau_3,\tau_4) + N^{-1}\Gamma(\tau_1,\tau_2,\tau_3,\tau_4)$$

$$\Gamma(\tau_1,\tau_2,\tau_3,\tau_4) =$$



$$-\,(3 \leftrightarrow 4)$$

Replica-diagonal effective action for $N \gg 1$

- Dynamic variables: Σ and G

$$\frac{\beta F}{N} = -\frac{1}{N}\overline{\ln Z} = -\frac{1}{N}\lim_{M\to 0}\frac{\overline{Z^M}}{M} \qquad (M \text{ is the number of replicas})$$

$$= \boxed{\begin{aligned} &-\ln \mathrm{Pf}(-\partial_\tau - \Sigma) \\ &+\frac{1}{2}\iint\left(\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{J^2}{q}\left|G(\tau_1,\tau_2)\right|^q\right)d\tau_1\,d\tau_2 \end{aligned}}$$

negligible
if $\beta J \gg 1$

- Stationary points are solutions of the Schwinger-Dyson equations. In particular, for $\beta = \infty$, we get

$$-G(\tau_1,\tau_2) = b^{\Delta}J^{-2\Delta}\left|\tau_1-\tau_2\right|^{-2\Delta}\operatorname{sgn}(\tau_1-\tau_2)$$

$$-\Sigma(\tau_1,\tau_2) = b^{1-\Delta}J^{2\Delta}\left|\tau_1-\tau_2\right|^{-2(1-\Delta)}\operatorname{sgn}(\tau_1-\tau_2)$$

where $\Delta = \frac{1}{q}$ and $b = \frac{1}{\pi}\left(\frac{1}{2}-\Delta\right)\tan(\pi\Delta)$

Emergent symmetries for $\beta J \gg 1$

- The effective action (neglecting the ∂_τ term) is invariant under

$$\boxed{\begin{aligned} G(\tau_1, \tau_2) \; &\longrightarrow \; G(f(\tau_1), f(\tau_2))\, f'(\tau_1)^\Delta f'(\tau_2)^\Delta \\ \Sigma(\tau_1, \tau_2) \; &\longrightarrow \; \Sigma(f(\tau_1), f(\tau_2))\, f'(\tau_1)^{1-\Delta} f'(\tau_2)^{1-\Delta} \end{aligned}}$$

For example, the transformation $f(\tau) = e^{2\pi i \tau/\beta} = z$ takes $G_\infty(z_1, z_2) = -b^\Delta J^{-2\Delta} (z_1 - z_2)^{-2\Delta}$ to the equilibrium Green function G_{eq} at finite β,

$$G_{\text{eq}}(\tau_1, \tau_2) = -b^\Delta \left(\frac{\beta J}{\pi} \sin \frac{\pi(\tau_1 - \tau_2)}{\beta} \right)^{-2\Delta} \quad \text{(for } 0 < \tau_1 - \tau_2 < \beta\text{)}$$

- The functions G_∞, G_{eq} are invariant under Möbius transformations preserving the unit circle:

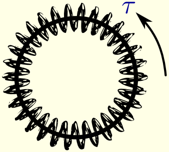
$$z \mapsto \frac{az + b}{cz + d}, \qquad \text{where} \quad z = e^{2\pi i \tau/\beta}$$

The reparametrization mode

- If we neglect the ∂_τ term, any Green function of the form

$$G_u(\tau_1,\tau_2) = G_{\mathrm{eq}}\big(u(\tau_1),u(\tau_2)\big)\,u'(\tau_1)^{\Delta}\,u'(\tau_2)^{\Delta}$$

is stationary and has the same energy as G_{eq}. Thus, the degeneracy space is $\mathrm{Diff}(S^1)/\,\mathrm{PSL}(2,\mathbb{R})$.



- The degeneracy is lifted due to a renormalized ∂_τ term, resulting in this effective free energy:

$$\boxed{F_{\mathrm{eff}} = -\frac{c}{4\pi^2}\int_0^\beta \{z,\tau\}\,d\tau, \qquad z(\tau) = \exp\!\big(i\,\tfrac{2\pi}{\beta}\,u(\tau)\big),}$$

where c is the specific heat, $c = \gamma\frac{N}{\beta J}$ for $\gamma \sim 1$, and $\{z,\tau\}$ is the Schwarzian derivative:

$$\{z,\tau\} = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2, \qquad z' = \frac{dz}{d\tau}.$$

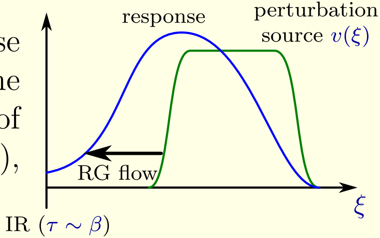
Renormalization theory (outline)

- The ∂_τ term has strong nonlinear effect on the Green function at $\tau \sim J^{-1}$, with some tails at longer times. Instead of solving the problem exactly, we will develop a renormalization theory to describe the IR tails.

- The actual perturbation $\Sigma \to \Sigma + \partial_\tau$ is replaced by a weak perturbation $\Sigma(\tau_1,\tau_2) \to \Sigma(\tau_1,\tau_2) + \sigma(\tau_1,\tau_2)$, where

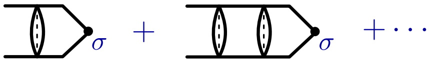
$$\sigma(\tau_1,\tau_2) = |\tau_1-\tau_2|^{2\Delta-1-\underset{\substack{\uparrow\\ \text{scaling}\\ \text{dimension}}}{\boxed{h}}}\,\mathrm{sgn}(\tau_1-\tau_2)\ \underbrace{v(\xi)}_{\substack{\text{window}\\ \text{function}}}, \qquad \xi = -\ln\Bigl(\tfrac{|\tau_1-\tau_2|}{\beta}\Bigr)$$

- One computes the linear response $\delta G(\tau_1,\tau_2)$. Typically, it stays in the same window. But for special values of the scaling dimension h (*resonances*), it leaks to the IR.



Renormalization theory (cont.)

$$\delta G = (K + K^2 + \cdots)\,\sigma =$$



- The resonances occur at the poles of $\dfrac{K(h)}{1-K(h)}$, where

$$K(h) = \frac{\varphi(2\Delta - h)\,\varphi(2\Delta + h - 1)}{\varphi(2\Delta - 2)\,\varphi(2\Delta + 1)}, \qquad \varphi(x) = \sqrt{\tfrac{2}{\pi}}\,\Gamma(x)\,\sin\tfrac{\pi x}{2}$$

- The first resonance is at $h = 2$.

- One can show that in the linear order, the corresponding perturbation σ is equaivalent to $\delta F \sim \int G(\tau,\tau)^q\,d\tau$.

Interpretation: $$G(\tau_1,\tau_2)^q \sim (\tau_1 - \tau_2)^{-2} + \underbrace{g(\tau)}_{G(\tau,\tau)^q}$$

The Schwarzian action

$$\boxed{F_{\text{eff}} = -\frac{c}{4\pi^2}\int \{f(\tau),\tau\}\,d\tau, \quad f(\tau) = \exp\!\Big(i\,\tfrac{2\pi}{\beta}\,u(\tau)\Big), \quad u:\,S^1 \to S^1}$$

- Derivation idea:
  Under the transformation $\tau \to f(\tau)$, the subleading term in the expansion $G(\tau_1,\tau_2)^q \sim (\tau_1-\tau_2)^{-2} + g(\tau)$ becomes

$$\tilde{g}(\tau) = f'(\tau)^2\, g(f(\tau)) + \{f(\tau),\tau\}.$$

- Expansion to the quadratic order in $\delta u(\tau) = u(\tau) - \tau$:

$$F_{\text{eff}} \approx -\frac{c}{2\beta} + \frac{c}{8\pi^2}\int \Big((\delta u'')^2 - \big(\tfrac{2\pi}{\beta}\big)^2(\delta u')^2\Big)\,d\tau$$

- Remaining degeneracy:

$$\delta u(\tau) = \alpha_0 + \alpha_+ e^{-2\pi i\tau/\beta} + \alpha_- e^{2\pi i\tau/\beta}$$

  These null modes do not affect the Green function though.

Summary

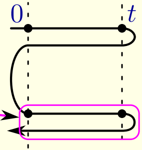
- In the large N limit, the SYK model is described by the replica-diagonal action $F[\Sigma, G]$.

- At low temperatures ($\beta J \gg 1$), there is a $\mathrm{Diff}(S^1)/\,\mathrm{PSL}(2,\mathbb{R})$ pseudo-Goldstone mode, which is described by the Schwarzian effective action F_{eff}.

- The Schwarzian action has three null modes, which may be understood as $\mathfrak{sl}(2,\mathbb{R})$ gauge symmetries. They do not affect any “physical” (i.e. Keldysh) correlators but are important for the calculation of OTO correlators.

In real time, $\quad \delta u(t) = \alpha_+ e^{\varkappa t} + \alpha_- e^{-\varkappa t}$,

$$\varkappa = \frac{2\pi}{\beta}$$



the null modes act on this part of the contour

Some further thoughts

- The Schwarzian action is local in τ and, therefore, describes coherent dynamics. Furthermore, for reparamerizations of this special form,

  $$\delta u(t) = \alpha_+(t)\, e^{2\pi t/\beta} + \alpha_-(t)\, e^{-2\pi t/\beta}, \quad \alpha_+, \alpha_- \text{ are slowly varying,}$$

  the Schwarzian action is similar to the adiabatic Berry phase.

- The OTO growth exponent $\varkappa$ has a small negative correction proportional to $(\beta J)^{-1}$ (Maldacena and Stanford, 2016). It can also be obtained from a subleading nonlocal term in the effective action:

  $$\sim \beta^{-1} J^{-2} \iint \frac{u'(\tau_1)\, u'(\tau_2)}{(\tau_1 - \tau_2)^4} \ln(J|\tau_1 - \tau_2|)\, d\tau_1\, d\tau_2.$$