Universal transport at the edge:

Disorder, interactions, and topological protection

Matthew S. Foster, Rice University

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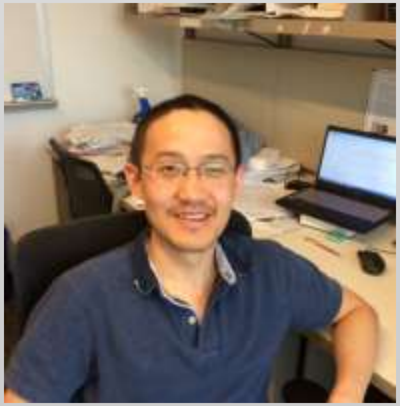
Universal transport coefficients at the edges of 2D topological insulators and 3D topological superconductors

Hong-Yi Xie, Yang-Zhi Chou, Heqiu Li, and Matthew S. Foster

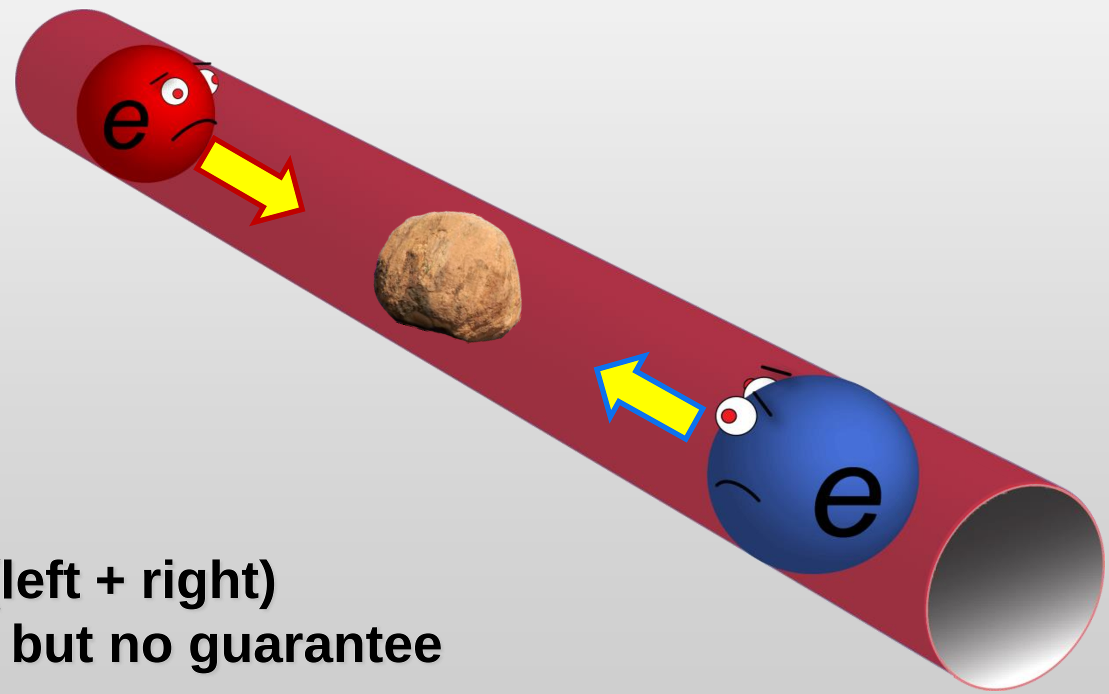


- PRL **116**, 086603 (2016)
- PRB **91**, 024203 (2015)
- PRB **89**, 165136 (2014)
- PRB **89**, 155140 (2014)
- PRL **109**, 246801 (2012)

Part I: Ballistic transport in 1D edge states of 2D topological insulators with Rashba spin-orbit coupling

How to protect a time-reversal invariant quantum wire ?

- Time-reversal invariant: need both left and right-movers!



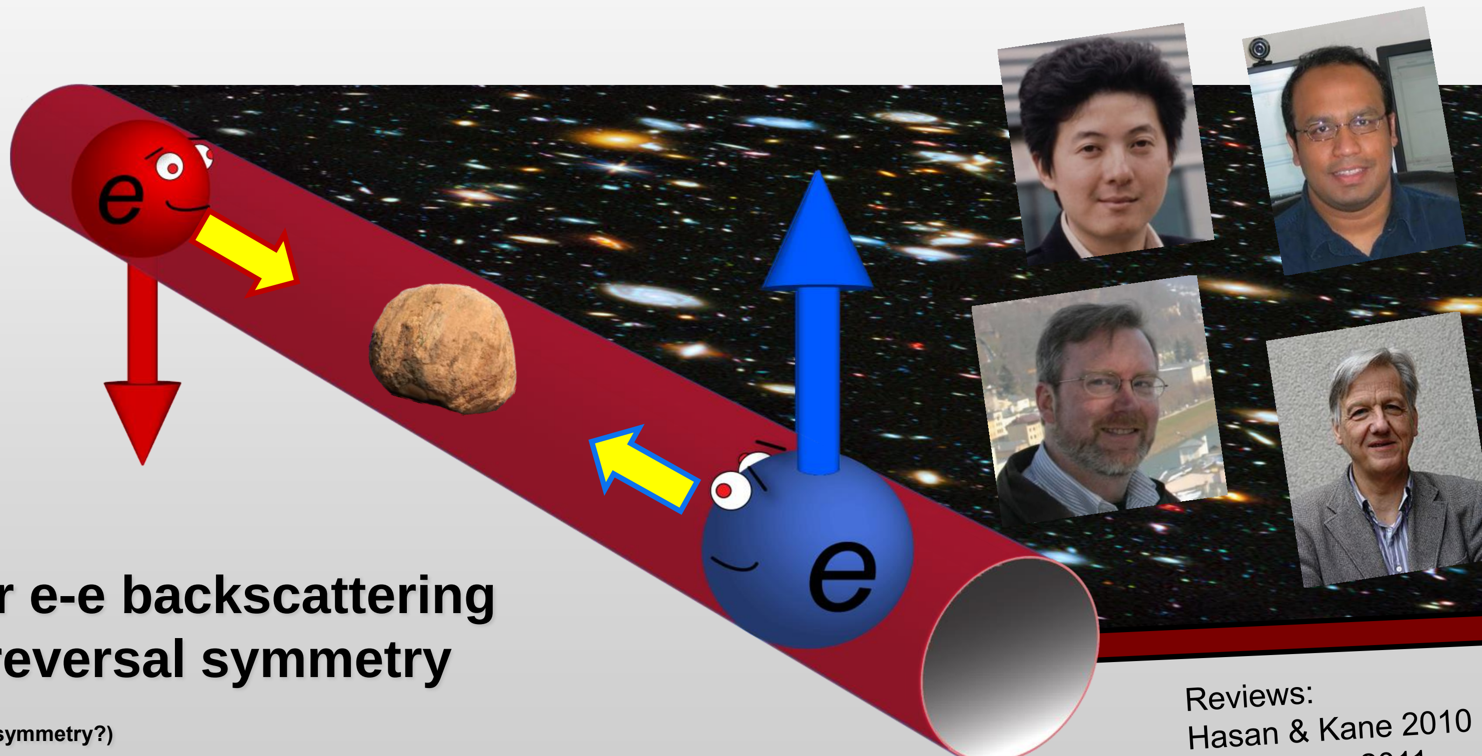
Total charge (left + right)
is conserved, but no guarantee
it will flow

(Anderson localization)

Helical quantum wire

- **Time-reversal invariant: need both left and right-movers!**
- **Solution: add spin!**

  (Left-up, Right-down: Perfect spin-momentum locking)



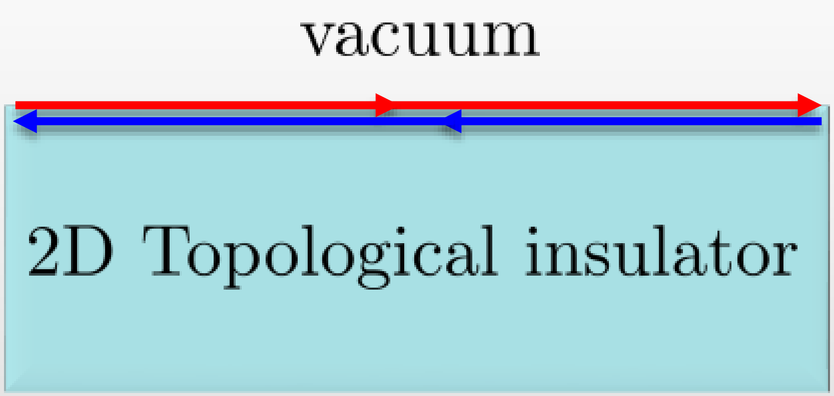
No impurity or e-e backscattering without time-reversal symmetry breaking (...or spin symmetry?)

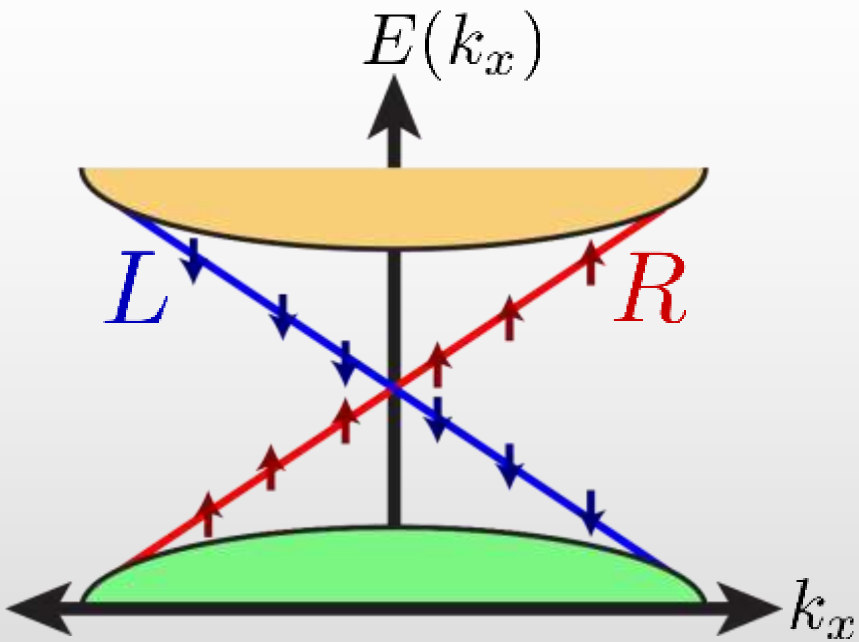
Reviews:
Hasan & Kane 2010
Qi & Zhang 2011

- **How to realize? At the edge of a 2D topological insulator!**

Quantum spin Hall 2D insulator

Ideal case: Invariant under time-reversal and spin S_z rotations





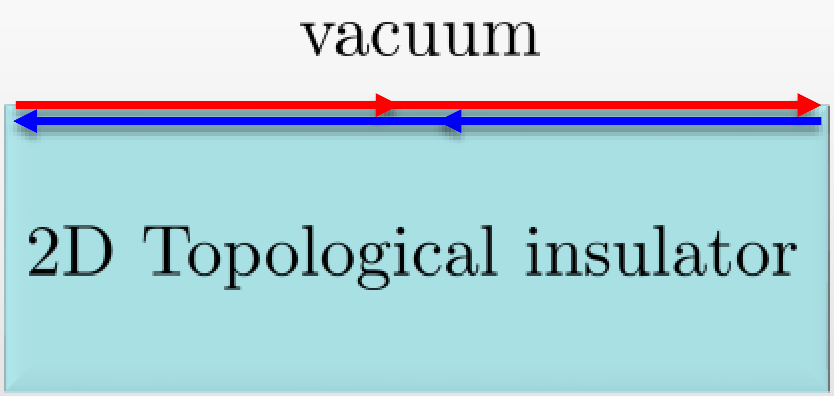
- **No backscattering whatsoever: perfect ballistic conduction!**
- $\sigma_{xx} = \frac{e^2}{h}$ **(per edge—Landauer formula)**
- **Weak (activated) contamination by bulk at non-zero, but low T**

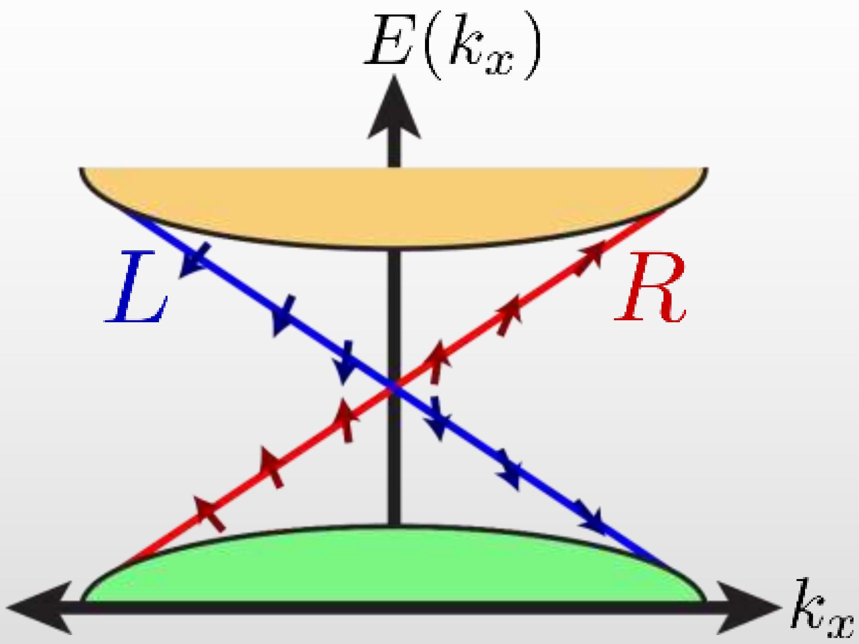
Kane and Mele 2005

2D Topological insulator with Rashba spin-orbit coupling

More realistic: Time-reversal only.
Rashba spin-orbit coupling (RSOC)
destroys spin symmetry

Kane and Mele 2005





- **Slow twisting of spin quantization axis,**

$$c_\uparrow(x) \simeq e^{ik_F x} R(x) - i\zeta e^{-ik_F x} \partial_x L(x)$$
$$c_\downarrow(x) \simeq e^{-ik_F x} L(x) - i\zeta^* e^{ik_F x} \partial_x R(x)$$

Xie, Li, Chou, Foster (2016)

- **Strength of RSOC:** $\zeta = 2k_F/k_0^2$

Schmidt, Rachel, von Oppen, Glazman (2012)

2D Topological insulator with Rashba spin-orbit coupling

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- **Density operator acquires backscattering part:**

$$\rho \simeq R^\dagger R + L^\dagger L - \left\{ i\zeta e^{-2ik_F x} \left[R^\dagger \partial_x L - (\partial_x R^\dagger) L \right] + \mathrm{H.c.} \right\}$$

2D Topological insulator with Rashba spin-orbit coupling

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Attributes of a generic helical edge (Luttinger) liquid:

1. **Potential disorder that couples to $\rho(x)$ induces random one-particle backscattering ("random RSOC")**

   Stroem, Johannesson, Japaridze (2010)
   Geissler, Crepin, Trauzettel (2014)

2. **Screened Coulomb interactions $\rho^2(x)$ include usual Luttinger interaction $R^\dagger R\, L^\dagger L$, plus (irrelevant) backscattering**

Effects of disorder

- **Single-particle Hamiltonian: 1D Dirac fermion** $\Psi(x) = \begin{bmatrix} R(x) \\ L(x) \end{bmatrix}$

$$H_0 = \int dx\, \Psi^\dagger\, \hat{h}\, \Psi$$

$$\boxed{\hat{h} = \hat{j}(x)\,(-i\partial_x) - \tfrac{i}{2}\partial_x \hat{j}(x) + V(x)}$$

- **Random current operator (due to disorder and RSOC)**

$$\hat{j}(x) = \boldsymbol{\gamma}(x) \cdot \hat{\boldsymbol{\sigma}}, \qquad \hat{\boldsymbol{\sigma}} = \left(\hat{\sigma}^1, \hat{\sigma}^2, \hat{\sigma}^3\right)$$

- **Components** $\boldsymbol{\gamma}(x) = \left[\xi_1(x), \xi_2(x), \mathrm{v_F}(x)\right]$

Edge transport with random Rashba spin-orbit coupling (RSOC)

$$\hat{h} = \hat{j}(x)\,(-i\partial_x) - \tfrac{i}{2}\partial_x \hat{j}(x) + V(x)$$

Schroedinger equation $\hat{h}\,\psi(x) = \varepsilon\psi(x)$

- **Re-write in terms of rescaled wavefunction,** $\varphi(x) \equiv \sqrt{\|\hat{j}(x)\|}\,\psi(x)$

$$(-i\partial_x)\,\varphi(x) = \hat{\mathcal{H}}_\varepsilon(x)\,\varphi(x), \quad \hat{\mathcal{H}}_\varepsilon(x) = \mathbf{b}(x)\cdot\hat{\boldsymbol{\sigma}}$$

Equivalent to time-dependent Schroedinger equation for a spin-1/2 moment in a fluctuating magnetic field!

$$i\partial_t\,|\varphi(t)\rangle = -\mathbf{B}(t)\cdot\hat{\boldsymbol{\sigma}}\,|\varphi(t)\rangle$$

$$|\varphi(t)\rangle = \alpha(t)\,|\!\uparrow\rangle + \beta(t)\,|\!\downarrow\rangle$$

Topological protection: Integrable spin-1/2 dynamics

Helical edge state with random Rasba spin-orbit coupling:
Fluctuating magnetic field has two components $\mathbf{B}(t) = \mathbf{B}_1(t) + \mathbf{B}_2(t)$

1. **First component: completely random (due to RSOC and disorder)**

$$\mathbf{B}_1(t) \equiv B_1(t)\,\hat{n}(t)$$

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$$\mathbf{B}_1(t) \equiv B_1(t)\,\hat{n}(t)$$

2. **Second component: slaved to the first**

$$\mathbf{B}_2(t) \equiv \frac{1}{2}\hat{n}(t) \times \partial_t \hat{n}(t)$$

$$\boxed{\hat{h} = \hat{j}(x)\,(-i\partial_x) - \frac{i}{2}\partial_x \hat{j}(x) + V(x)}$$

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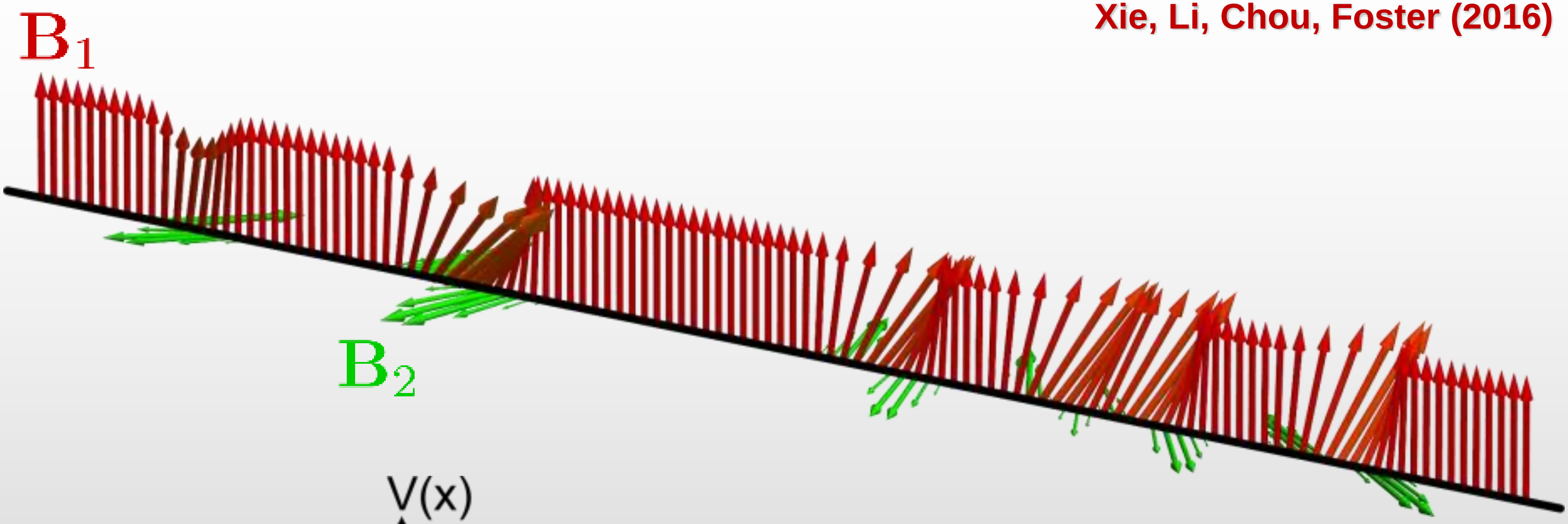
Perfect "adiabatic counterterm"!

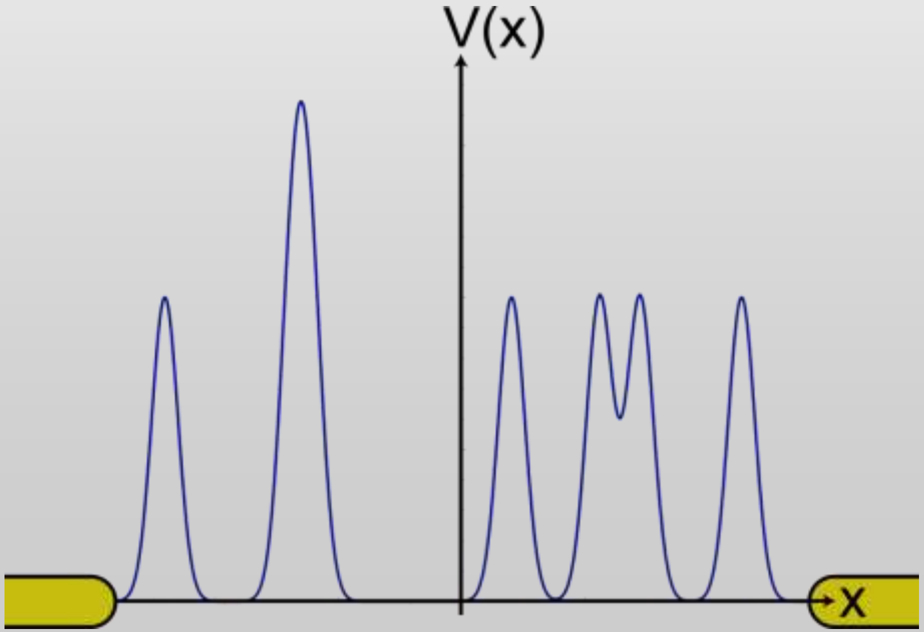
Can show that a state initially aligned along $\mathbf{B}_1(t)$ **exactly follows it for all subsequent times!**

Xie, Li, Chou, Foster (2016)

Topological protection: Integrable spin-1/2 dynamics

Xie, Li, Chou, Foster (2016)





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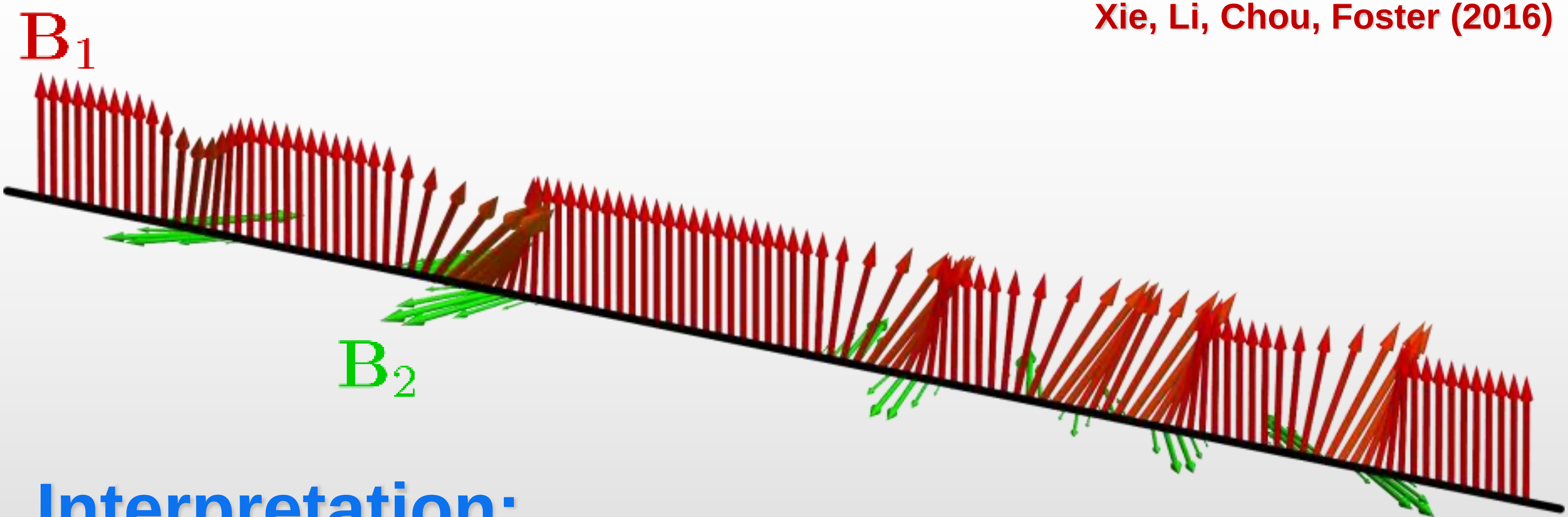
Xie, Li, Chou, Foster (2016)



Interpretation:

Right-mover entering from ideal left lead exits as right-mover through right lead with 100% probability.

Perfect transmission! $\sigma_{xx} = \dfrac{e^2}{h}$

Explicit mechanism for ballistic transport predicted by Kane/Mele 2005

Random RSOC and Luttinger interactions: Still protected??

- **Generic edge theory with disorder, interactions***

$$H = \int dx \left\{ \Psi^\dagger \left[\hat{j}(x) \left(-i\partial_x\right) - \tfrac{i}{2}\partial_x \hat{j}(x) + V(x) \right] \Psi + U(x) \, \left(\Psi^\dagger \Psi\right)^2 \right\}$$

- **Transformation using the (integrable) transfer matrix:**

$$\Phi(x) \equiv \sqrt{\|\hat{j}(x)\|/\mathrm{v_F}} \;\; \hat{\mathsf{T}}_0^\dagger(x, -L/2) \, \Psi(x)$$

- **Homogeneous kinetic term, inhomogeneous interaction:**

$$H = \int dx \left\{ \Phi^\dagger \left[v_F \hat{\sigma}^3 \left(-i\partial_x\right) \right] \Phi + \tilde{U}(x) \, \left(\Phi^\dagger \Phi\right)^2 \right\}$$

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Equivalent to a free boson! **Xie, Li, Chou, Foster (2016)**

$$S = \frac{1}{2}\int dt\, dx \left[\frac{1}{\mathsf{v}_{\mathrm{c}}(x) K(x)} \left(\partial_t \theta\right)^2 - \frac{\mathsf{v}_{\mathrm{c}}(x)}{K(x)} \left(\partial_x \theta\right)^2 \right]$$

Guarantees ballistic conduction through ideal leads $\sigma_{xx} = \dfrac{e^2}{h}$

Maslov & Stone (1995), Ponomarenko (1995), Safi & Schultz (1995)

*** Neglecting irrelevant umklapp interactions, which give corrections at finite temperature.**

Schmidt *et al.* 2012; Kainaris *et al.* 2014; Chou, Levchenko, and Foster 2015

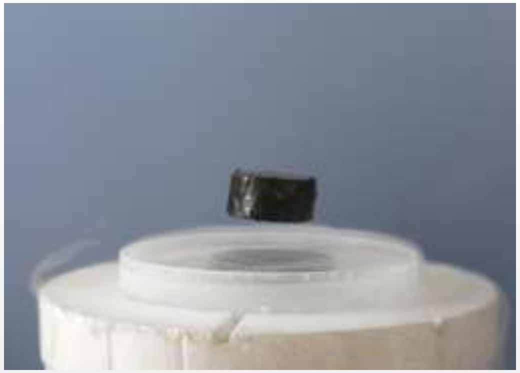
Part II: Quantized spin and thermal transport

at the surface of a 3D topological superconductor

Topological Superconductor: Gapped bulk, Majorana fluid boundary

Superconductivity

Collective motion of loosely bound electron pairs at low temperatures



Mai-Linh Doan, Wikipedia

- **Superfluidity: Electrical resistance is zero**
- **No heat or spin transport in the superfluid**
- **Topological superconductor:**
  Theorized to possess a charge neutral surface fluid of unpaired "Majorana" fermions

Spaceballs the Movie

"New" idea: 3D Bulk topological superconductivity

- **Integer-valued winding number $\nu \in \mathbb{Z}$**
- **2D Majorana surface fluid**
- **Transport properties?**

Schnyder, Ryu, Furusaki, Ludwig 2008; Kitaev 2009

Experimental realizations

- **Helium 3B (neutral topological superfluid)**

  Volovik 1988

- **$Cu_xBi_2Se_3$?, Cd_3As_2 ?, LuPdBi ?**

Fu and Berg 2010;
L. Wray, Z. Hasan *et al*. 2010;
G.-q. Zheng *et al*. 2015
…

Topological superconductor surface state Majorana fluid

For a 3D topological superconductor with bulk winding number ν, what do the Majorana surface states "look like?"

Spaceballs the Movie

Topological superconductor surface state Majorana fluid

A lot like graphene!

- **Unpaired surface Majorana fermion quasiparticles**
- **$|\nu| = 2k$ "colors," $k = (1,2,3,\ldots)$ (class CI)**

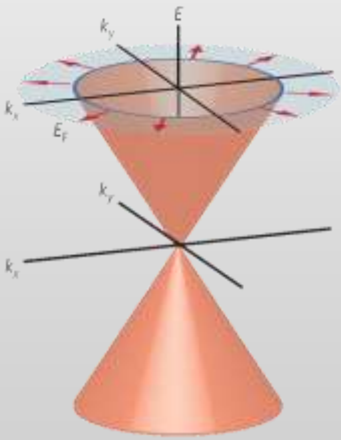
Low energy surface Andreev state Hamiltonian:

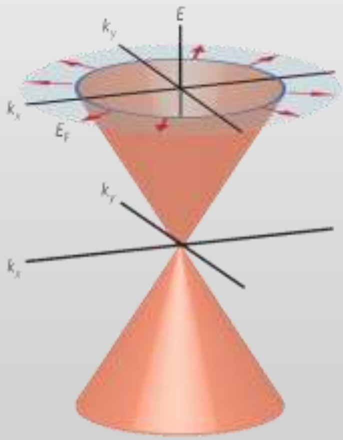
$$H = \int d^2\mathbf{r}\,\Psi^\dagger\left(-i\hat{\sigma}^1\partial_x - i\hat{\sigma}^2\partial_y\right)\Psi \equiv \Psi^\dagger\hat{h}\Psi$$

Majorana fermion carries
Pseudospin (σ) and
Color (κ) indices

$$\Psi_\kappa = \begin{bmatrix} c_{\uparrow,\kappa} \\ c^\dagger_{\downarrow,\bar{\kappa}} \end{bmatrix}$$

$$1 \leq \kappa \leq |\nu|$$





- **"Anomalous" chiral symmetry (= *physical time-reversal*):** $-\hat{\sigma}^3\hat{h}\hat{\sigma}^3 = \hat{h}$

Schnyder, Ryu, Furusaki, Ludwig 08; Bernard, LeClair 02

Topological superconductor surface state Majorana fluid

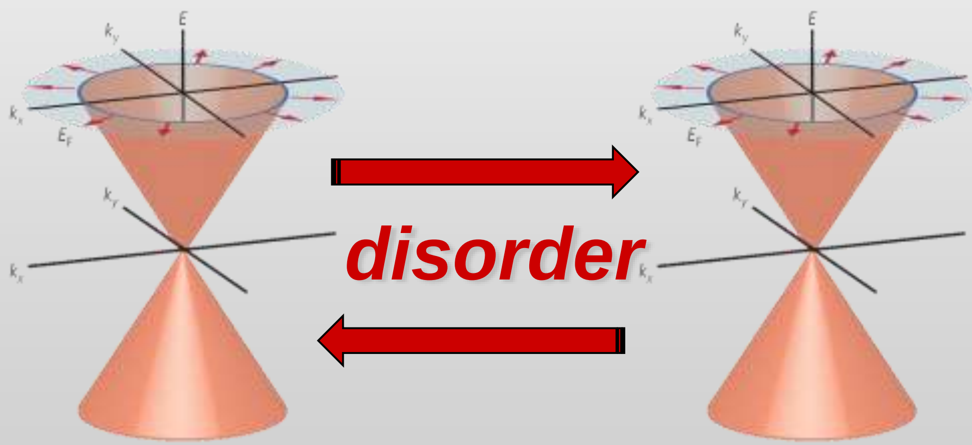
Effects of disorder

- **Junk is unavoidable at the surface!**
- ***Any* non-magnetic (time-reversal preserving) surface perturbation: intercolor vector potential $\hat{t}^i_\kappa \mathbf{A}_i(\mathbf{r})$!**

$$H = \int d^2\mathbf{r}\, \Psi^\dagger \left(-i\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nabla} + \mathbf{A}_i \cdot \boldsymbol{\sigma}\, \hat{t}^i_\kappa\right) \Psi = H_0 + \int d^2\mathbf{r}\, \left(J^i_\kappa \bar{A}_i + \bar{J}^i_\kappa A_i\right)$$

Sources of $\hat{t}^i_\kappa \mathbf{A}_i(\mathbf{r})$:

- **Impurities, vacancies**
- **External electric fields**
- **Edge, corner, dislocation potentials**



"Quenched" 2+1-D QCD: Dirac fermions in a sea of frozen gauge fluctuations

Schnyder, Ryu, Furusaki, Ludwig 08; Bernard, LeClair 02

Majorana fluid transport

Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. **Multiple scattering off of impurities (weak localization)**

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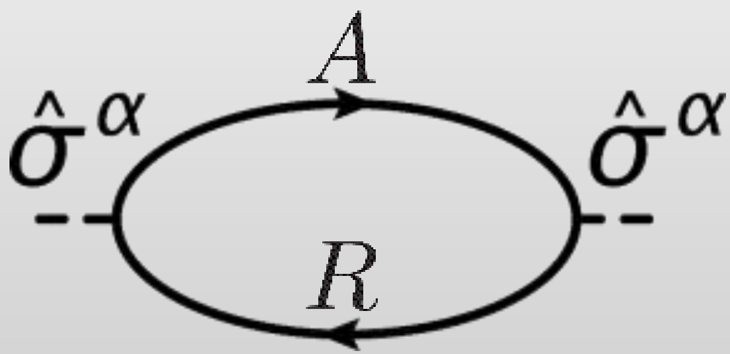
Kubo formula for dc spin conductivity (class CI or AIII):

$$\sigma = \frac{1}{8\pi L^d} \int_{\mathbf{r}_1,\mathbf{r}_2} \mathrm{Re}\left\{\mathrm{Tr}\left[\hat{\sigma}^\alpha \hat{G}^{(A)}(0;\mathbf{r}_1,\mathbf{r}_2)\hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r}_2,\mathbf{r}_1) - \hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r}_1,\mathbf{r}_2)\hat{\sigma}^\alpha \hat{G}^{(R)}(0;\mathbf{r}_2,\mathbf{r}_1)\right]\right\}$$



Components:
Retarded, advanced Green's functions

$$\hat{G}^{(R,A)}(\varepsilon;\mathbf{r}_1,\mathbf{r}_2) \equiv \langle \mathbf{r}_1 | \frac{1}{\varepsilon \pm i\eta - \hat{h}} | \mathbf{r}_2 \rangle \qquad \hat{h} = -i\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nabla} + \mathbf{A}_i(\mathbf{r}) \cdot \boldsymbol{\sigma}\, \hat{t}^i_\kappa$$

Surface Majorana fluid can carry spin or heat

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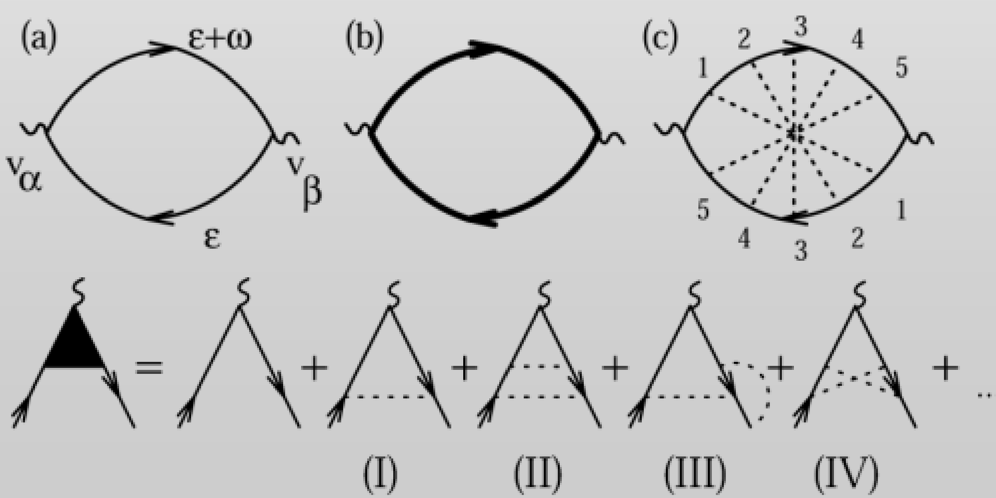
"Normal life:" average over disorder, sum many diagrams with many impurity lines



See, e.g., Altland and Simons for a review

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Anomalous form of time-reversal symmetry

Retarded, advanced interchangable: $-\hat{\sigma}^3 \hat{G}^{(A)}(\varepsilon;\mathbf{r}_1,\mathbf{r}_2)\hat{\sigma}^3 = \hat{G}^{(R)}(-\varepsilon;\mathbf{r}_2,\mathbf{r}_1)$

$\Rightarrow$ $\sigma = -\frac{1}{\pi} \lim_{\mathbf{r}\to\mathbf{r}'} \operatorname{Im}\left\{\operatorname{Tr}\left[(\mathbf{r}-\mathbf{r}')\cdot\hat{\boldsymbol{\sigma}}\,\hat{G}^{(R)}(0;\mathbf{r},\mathbf{r}')\right]\right\} = \frac{|\nu|}{2\pi^2}$ **2+0-D Chiral anomaly**

Universal spin, thermal conductivity, neglecting interactions

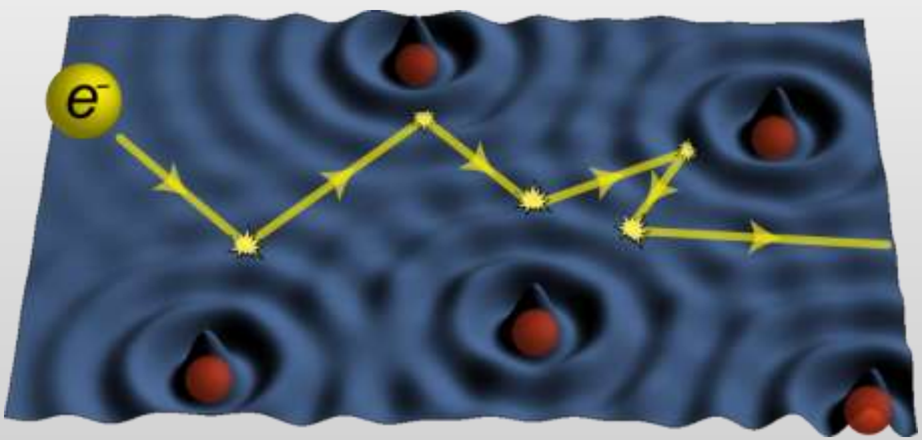
$$\sigma_s = \frac{|\nu|}{\pi h}\left(\frac{\hbar}{2}\right)^2, \qquad \kappa = \frac{|\nu|}{\pi h}\frac{\pi^2 k_B^2 T}{3}$$

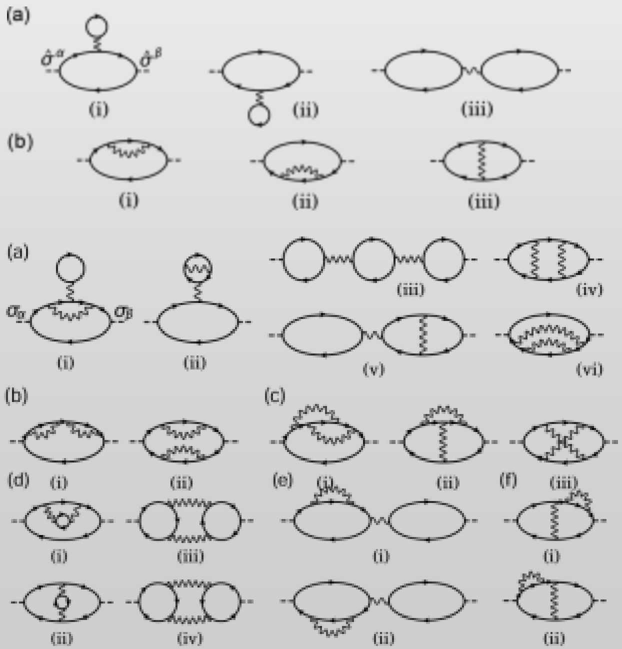
Ludwig, Fisher, Shankar, Grinstein (1994)
Tsvelik (1995)
Ostrovsky, Gornyi, Mirlin (2006)
Foster, Xie, and Chou (2014)

Surface Majorana fluid can carry spin or heat

In 2D, wave interference dominates transport; quantum conductance corrections due to

1. Multiple scattering off of impurities (weak localization)
2. **Scattering off of impurity-induced density Friedel oscilations (Altshuler-Aronov corrections, short-ranged interactions)**





Altshuler and Aronov 1985 (Review)
Aleiner, Altshuler, and Gershenson 1999 (Review)
Zala, Narozhny, and Aleiner 2001

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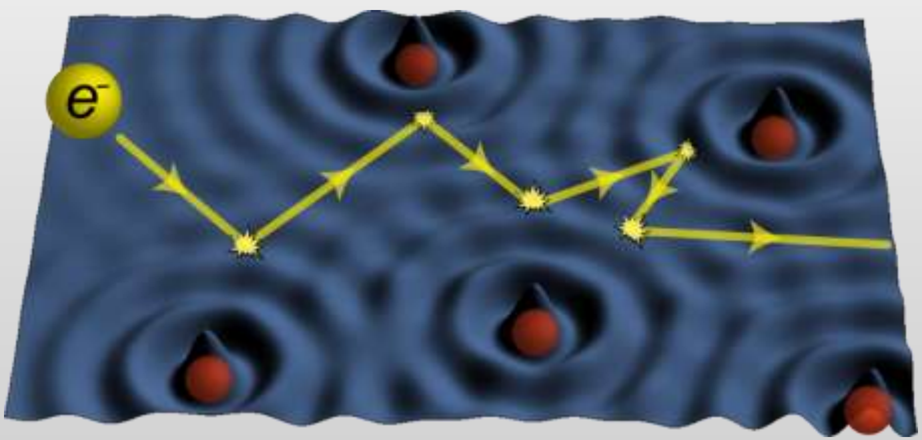
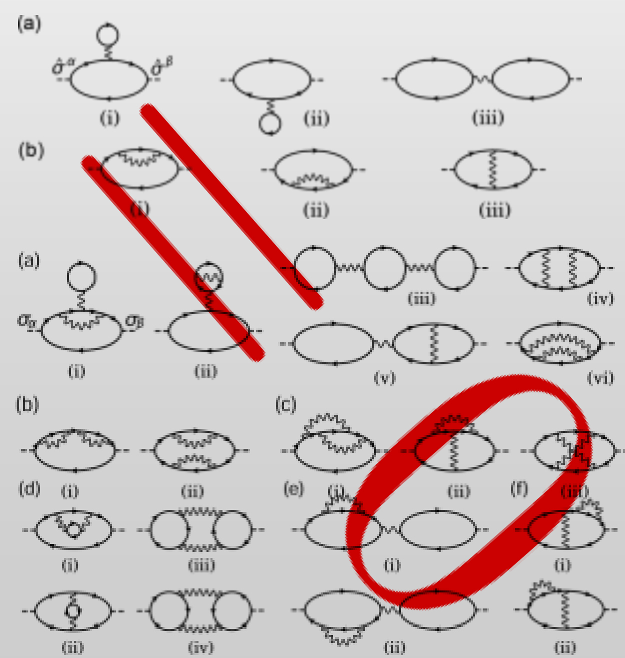




1. Anomalous time-reversal symmetry:

$$-\hat{\sigma}^3 \hat{G}^{(A)}(\varepsilon;\mathbf{r_1},\mathbf{r_2})\hat{\sigma}^3 = \hat{G}^{(R)}(-\varepsilon;\mathbf{r_2},\mathbf{r_1})$$

2. Spin U(1) Ward identity:

$$\int_{\mathbf{r}} \hat{G}^{(R)}(\varepsilon;\mathbf{x_1},\mathbf{r})\hat{\sigma}^{\alpha}\hat{G}^{(R)}(\varepsilon;\mathbf{r},\mathbf{x_2}) = -i\left(\mathbf{x_1}-\mathbf{x_2}\right)^{\alpha}\hat{G}^{(R)}(\varepsilon;\mathbf{x_1},\mathbf{x_2})$$

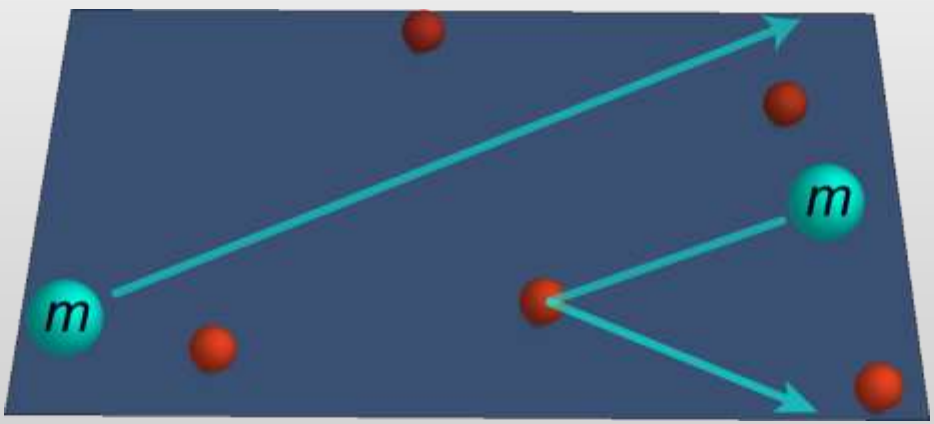
Xie, Chou, and Foster (2015)

Majorana fluid transport

Surface Majorana fluid can carry spin or heat

Statements for spin conductance in every fixed realization of disorder:

1. No Altshuler-Aronov corrections to first order in generic short-ranged interactions (spin triplet, Cooper pairing): classes CI, AIII
2. No Altshuler-Aronov to second order (two loops): class AIII



Anomalous time-reversal symmetry:

No Majorana "density" (mass, spin, color) can ripple (or become non-zero)!

Universal spin, thermal* conductivities!

$$\sigma_s = \frac{|\nu|}{\pi h}\left(\frac{\hbar}{2}\right)^2$$

$$\kappa = \frac{|\nu|}{\pi h}\frac{\pi^2 k_B^2 T}{3}$$

Xie, Chou, and Foster (2015)

Majorana fluid transport redux: Large winding number expansion

- **Clean, non-interacting fermions:**

$$S = \sum_{\omega_n} \int d^2\mathbf{r}\, \bar{\Psi}_{\kappa,a}(\omega_n) \left(-i\omega_n - i\hat{\sigma}^1\partial_x - i\hat{\sigma}^2\partial_y\right) \Psi_{\kappa,a}(\omega_n)$$

- $\kappa \in 1, \ldots, 2k$ (CI) k (AIII, DIII) number of colors (valleys); $k \propto$ winding number
- $a \in 1, \ldots n$ "replica" index, used for disorder-averaging, $n \to 0$ in the end
- $\beta\omega_n/2\pi \in \mathbb{Z} + 1/2$ Matsubara frequency

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$$S = \sum_{\omega_n} \int d^2\mathbf{r}\, \bar{\Psi}_{\kappa,a}(\omega_n) \left(-i\omega_n - i\hat{\sigma}^1\partial_x - i\hat{\sigma}^2\partial_y\right) \Psi_{\kappa,a}(\omega_n)$$

  - $\kappa \in 1,\ldots,2k$ (CI) k (AIII, DIII) number of colors (valleys); $k \propto$ winding number
  - $a \in 1,\ldots n$ "replica" index, used for disorder-averaging, $n \to 0$ in the end
  - $\beta\omega_n/2\pi \in \mathbb{Z} + 1/2$ Matsubara frequency

- **Non-abelian bosonization in 2+1-D: (frequency is an index)**

$$\Psi_{\kappa,a}(\omega_n,\mathbf{r})\, \bar{\Psi}_{\kappa',b}(\omega'_n,\mathbf{r}) \sim Q_{\kappa,a\;\kappa',b}(\omega_n,\omega'_n;\mathbf{r}), \qquad \hat{Q}^\dagger(\mathbf{r})\,\hat{Q}(\mathbf{r}) = \hat{1}$$

$$S = \frac{1}{8\pi}\int_{\mathbf{r}} \mathrm{Tr}\left[\partial_\mu \hat{Q}^\dagger \partial_\mu \hat{Q}\right] + S_{\mathrm{WZNW}} - \eta \int_{\mathbf{r}} \mathrm{Tr}\left[\hat{\omega}_N \left(\hat{Q} + \hat{Q}^\dagger\right)\right]$$

E.g., chapter 15 of the "big yellow" CFT book Di Francesco, Mathieu, Senechal

Majorana fluid transport redux: Large winding number expansion

- **Clean, non-interacting fermions:**

$$S = \sum_{\omega_n} \int d^2\mathbf{r}\, \bar{\Psi}_{\kappa,a}(\omega_n) \left(-i\omega_n - i\hat{\sigma}^1 \partial_x - i\hat{\sigma}^2 \partial_y\right) \Psi_{\kappa,a}(\omega_n)$$

$$\Psi_{\kappa,a}(\omega_n, \mathbf{r})\, \bar{\Psi}_{\kappa',b}(\omega_n', \mathbf{r}) \sim Q_{\kappa,a\;\kappa',b}(\omega_n, \omega_n'; \mathbf{r}), \qquad \hat{Q}^\dagger(\mathbf{r})\, \hat{Q}(\mathbf{r}) = \hat{1}$$

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- **Disorder: relevant perturbation, color sector is massive ("localizes")**

$$S = \frac{k}{8\pi} \int_{\mathbf{r}} \mathrm{Tr}\left[\partial_\mu \hat{Q}^\dagger \partial_\mu \hat{Q}\right] + k S_{\mathrm{WZNW}} - \eta \int_{\mathbf{r}} \mathrm{Tr}\left[\hat{\omega}_N \left(\hat{Q} + \hat{Q}^\dagger\right)\right]$$

- **Technically justified by conformal embedding $SO(4nk)_1 \Rightarrow Sp(2n)_k$**

Majorana fluid transport redux: Large winding number expansion

Effective field theory: 2+1-D WZNW Finkelstein Non-linear sigma model

- **Quenched disorder:** Exact treatment via non-abelian bosonization, conformal embedding $SO(4nk)_1 \Rightarrow Sp(2n)_k$
- **Interactions:** Controlled in large winding number limit $k = |\nu|/2 \gg 1$

$$S = \frac{1}{8\pi\lambda} \int_{\mathbf{r}} \mathrm{Tr}\left[\partial_\mu \hat{Q}^\dagger \partial_\mu \hat{Q}\right] + k S_{\mathrm{WZNW}} - \eta \int_{\mathbf{r}} \mathrm{Tr}\left[\hat{\omega}_N \left(\hat{Q} + \hat{Q}^\dagger\right)\right]$$

$$- \Gamma_t \sum_a \int_{\tau,\mathbf{r}} \left(\mathrm{Tr}_s \left\{\hat{\vec{S}}\left[\hat{Q}_{aa}(\tau,\tau) + \hat{Q}^\dagger_{aa}(\tau,\tau)\right]\right\}\right)^2$$

$$- \Gamma_c \sum_a \int_{\tau,\mathbf{r}} \left\{\mathrm{Tr}_s \left[\hat{Q}_{aa}(\tau,\tau) - \hat{Q}^\dagger_{aa}(\tau,\tau)\right]\right\}^2$$

c.f. Finkelstein 1983

- **Spin (CI, AIII) or heat resistance (DIII) encoded in** $\lambda = 1/k$

Majorana fluid transport redux: Large winding number expansion

Effective field theory: 2+1-D WZNW Finkelstein Non-linear sigma model

- Quenched disorder: Exact treatment via non-abelian bosonization, conformal embedding $SO(4nk)_1 \Rightarrow Sp(2n)_k$
- Interactions: Controlled in large winding number limit $k = |\nu|/2 \gg 1$
- Spin (CI, AIII) or heat resistance (DIII) encoded in λ

- One-loop RG equations for λ: $(\gamma_{s,c} \equiv 4\Gamma_{s,c}/\pi\eta)$

Xie, Chou, Foster (2015)

$$
\begin{aligned}
\text{CI:} \quad & d\lambda/dl = \lambda^2\left[1-(k\lambda)^2\right]\left[1+\mathcal{J}(\gamma_s,\gamma_c)\right], \\
\text{AIII:} \quad & d\lambda/dl = \lambda^2\left[1-(k\lambda)^2\right]\mathcal{I}(\gamma_s,\gamma_c), \\
\text{DIII:} \quad & d\lambda/dl = -\lambda^2\left[1-(k\lambda)^2\right]\left[2+\mathcal{K}(\gamma_c)\right].
\end{aligned}
$$

$\Rightarrow$ All corrections (incl Altshuler-Aronov) vanish for $\lambda = 1/k$!

Majorana fluid transport redux: Large winding number expansion

$$\begin{aligned}
\text{CI:} \quad & \mathcal{J}(\gamma_{\mathrm{s}}, \gamma_{\mathrm{c}}) = 3\left[1 + \tfrac{1-\gamma_{\mathrm{s}}}{\gamma_{\mathrm{s}}} \ln\left(1-\gamma_{\mathrm{s}}\right)\right] - \tfrac{1}{4}\mathcal{K}(\gamma_{\mathrm{c}}), \\
\text{AIII:} \quad & \mathcal{I}(\gamma_{\mathrm{s}}, \gamma_{\mathrm{c}}) = 2\left[1 + \tfrac{1-\gamma_{\mathrm{s}}}{\gamma_{\mathrm{s}}} \ln\left(1-\gamma_{\mathrm{s}}\right)\right] - \tfrac{1}{2}\mathcal{K}(\gamma_{\mathrm{c}}), \\
\text{DIII:} \quad & \mathcal{K}(\gamma_{\mathrm{c}}) = 2e^{-1/\gamma_{\mathrm{c}}}\left[E_i\left(\tfrac{1}{\gamma_{\mathrm{c}}} + \ln 2\right) - E_i\left(\tfrac{1}{\gamma_{\mathrm{c}}}\right)\right] \\
& = 2\gamma_{\mathrm{c}} + \boldsymbol{O}\left(\gamma_{\mathrm{c}}^2\right).
\end{aligned}$$

- **One-loop RG equations for λ:** $\quad (\gamma_{\mathrm{s,c}} \equiv 4\Gamma_{s,c}/\pi\eta)$

Xie, Chou, Foster (2015)

$$\begin{aligned}
\text{CI:} \quad & d\lambda/dl = \lambda^2\left[1-(k\lambda)^2\right]\left[1+\mathcal{J}(\gamma_{\mathrm{s}}, \gamma_{\mathrm{c}})\right], \\
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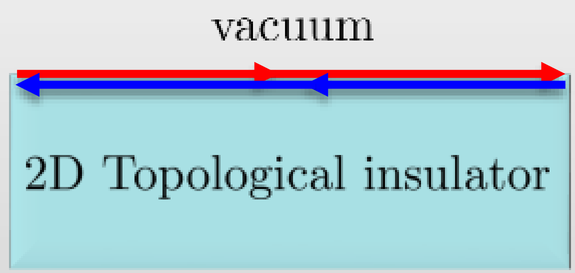
\end{aligned}$$

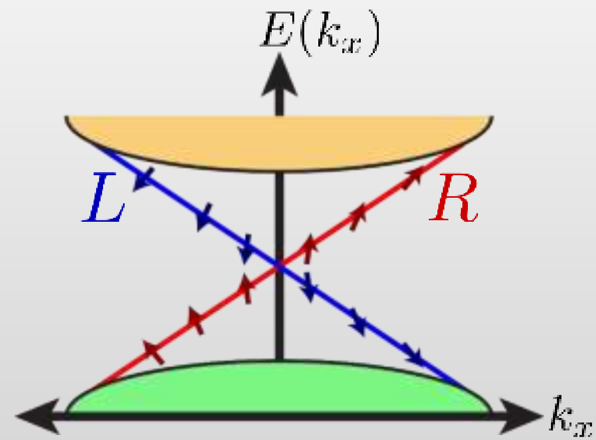
➡ **All corrections (incl Altshuler-Aronov) vanish for $\lambda = 1/k$!**

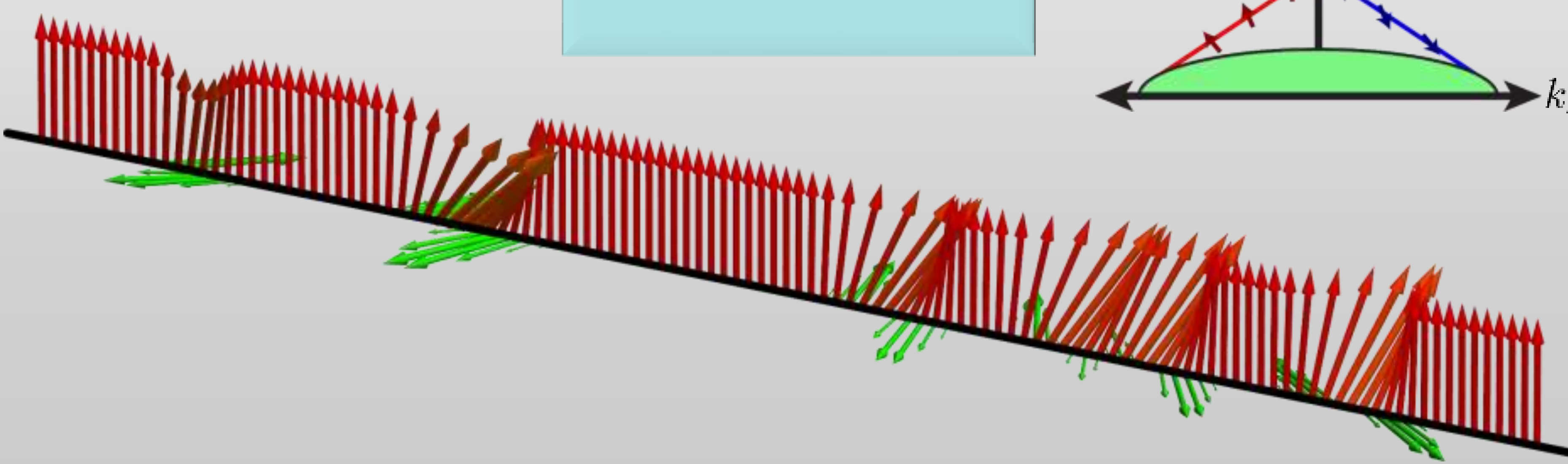
Summary

1D Helical edge states of 2D topological insulators

- **Rashba spin-orbit coupling ➡ backscattering disorder, interactions**
- **"Random Rashba" disorder = integrable spin-1/2 dynamics, no backscattering through ideal leads**
- **Unchanged by Luttinger liquid interactions**

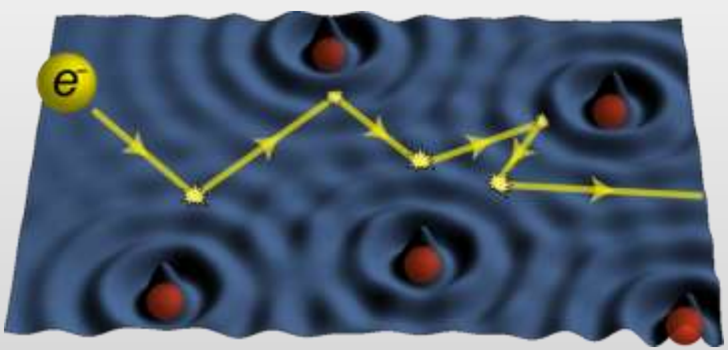


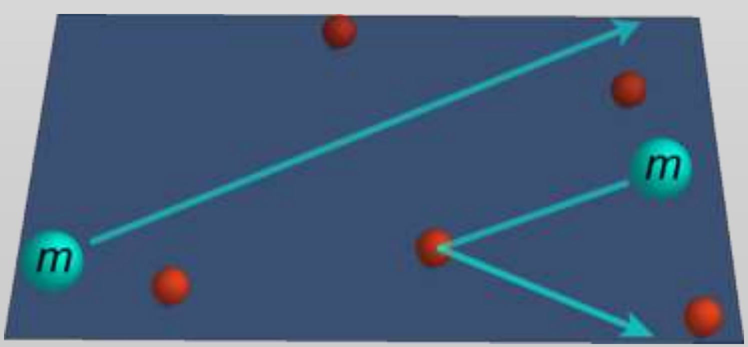


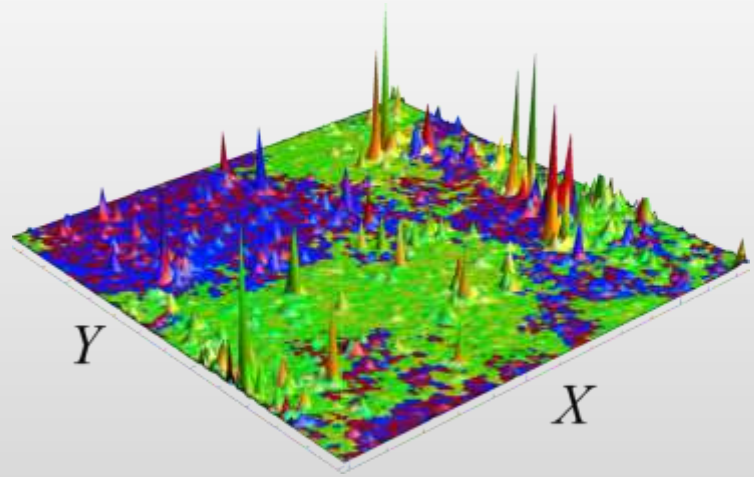


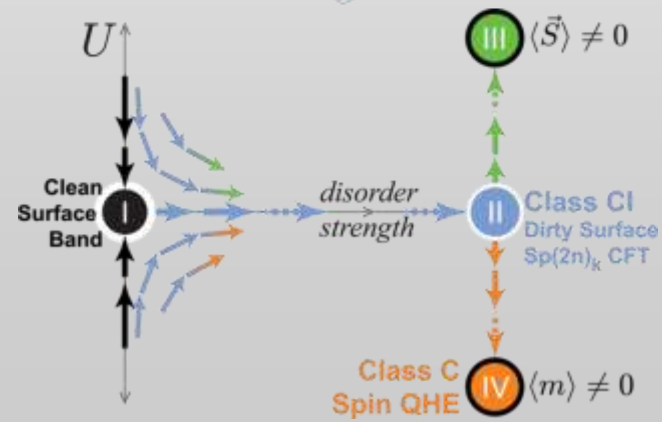
2D Majorana liquid theory

- **Surface states of a bulk topological superconductor**
- **Universal transport coefficients encode bulk winding number**
- **Combined effects of disorder and interactions can lead to instabilities**









Summary

2D Majorana liquid theory

- **Surface states of a bulk topological superconductor**
- **Universal transport coefficients encode bulk winding number**
- **Combined effects of disorder and interactions can lead to instabilities**

3D Topological superconductivity: close analog of the integer quantum Hall effect

- **Is there a fractional analog? (Bulk with topological order; gapless surface fluid with fractionalized transport coefficients)**
- **What about gapless (nodal) “topological” superconductor surface states?**

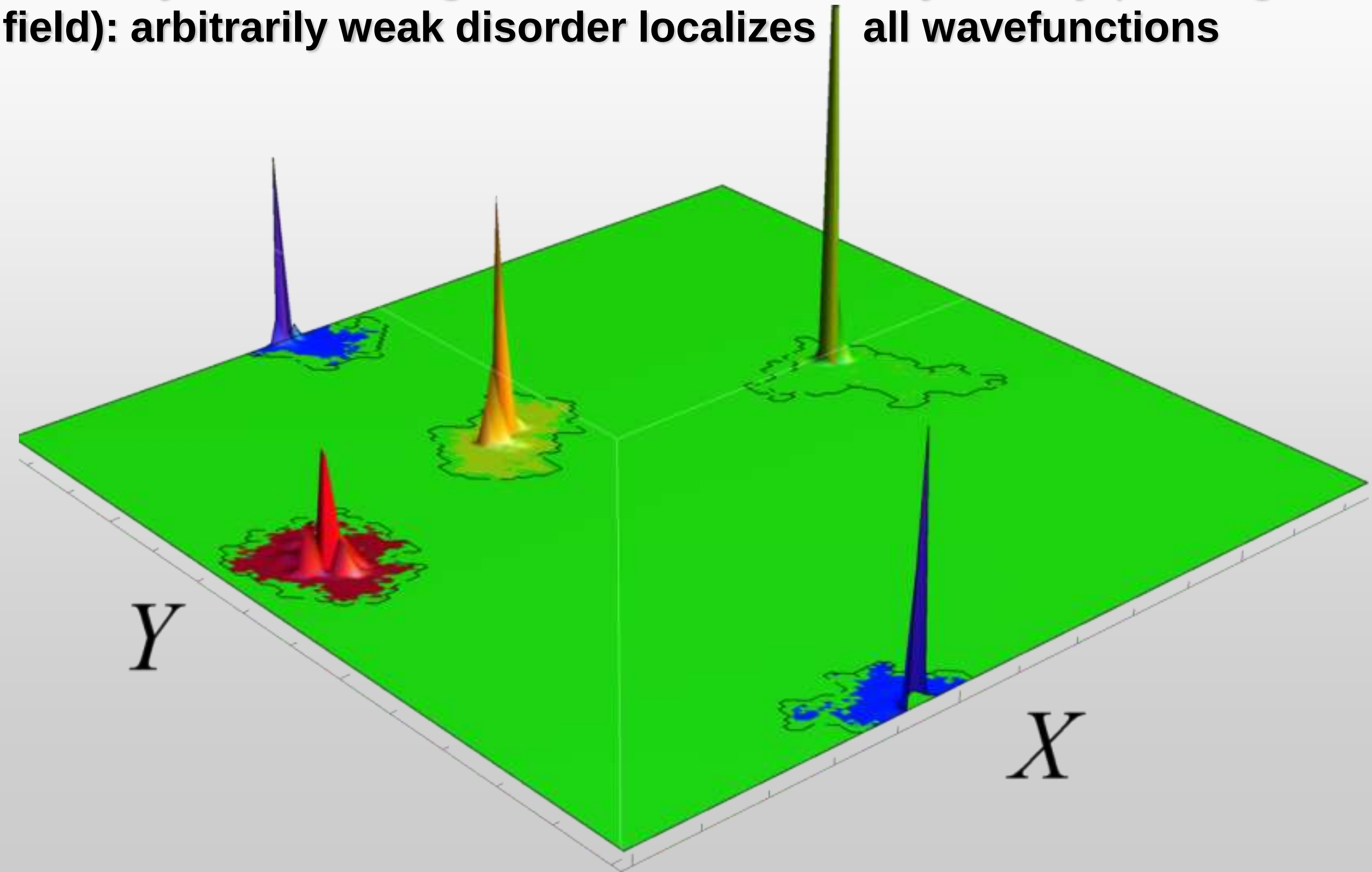
Sato 2006
Beri 2010
Schnyder and Ryu 2011
Matsuura, Chang, Schnyder, Ryu 2013
Zhao and Wang 2013

- **Materials?**

Majorana fluid wavefunctions

Universal transport: Disorder has no effect?

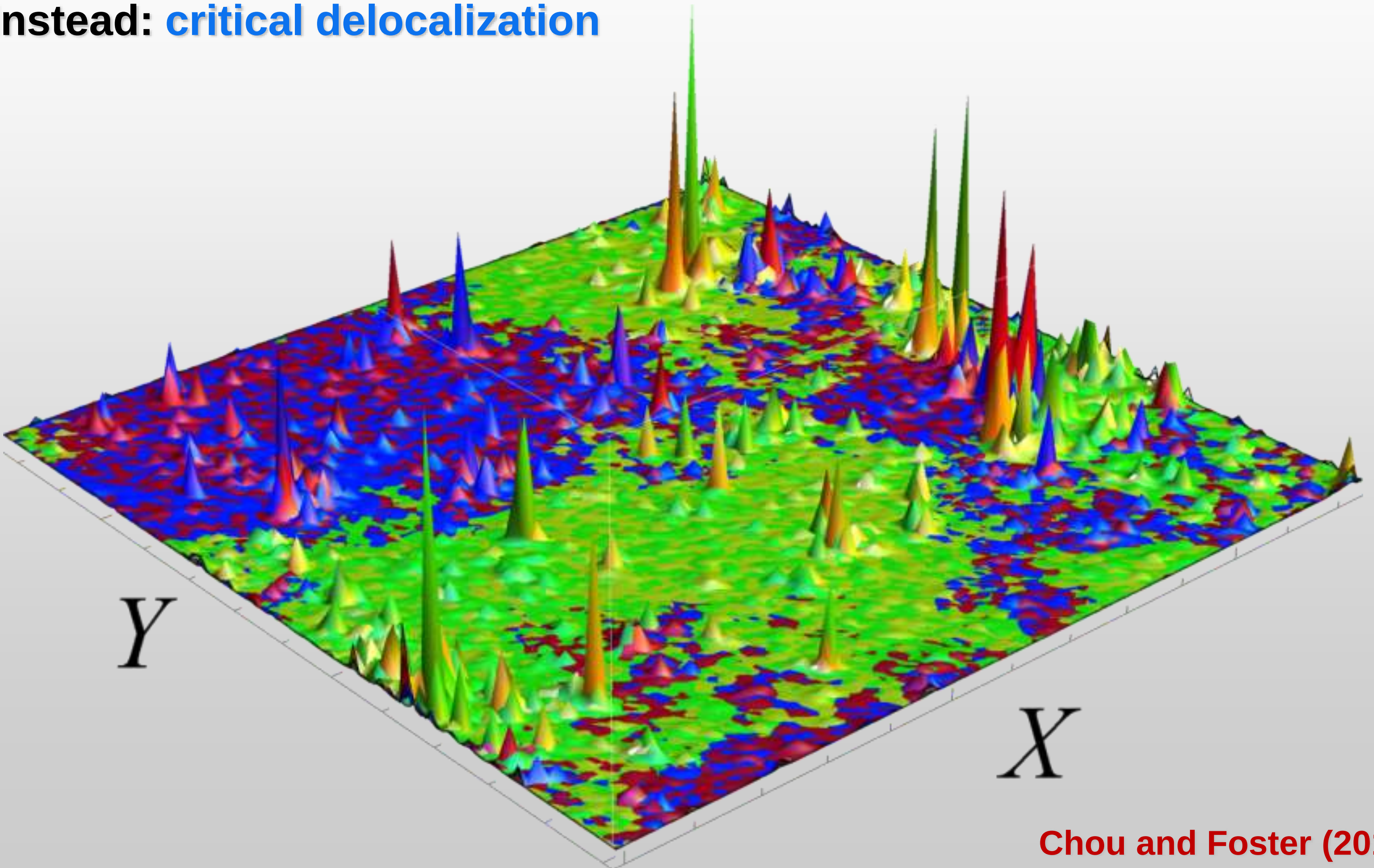
Ordinary 2D electron gas with time reversal symmetry (no magnetic field): arbitrarily weak disorder localizes all wavefunctions



Majorana fluid wavefunctions

Universal transport: Disorder has no effect?

Surface Majorana states cannot be localized (topological protection).
Instead: critical delocalization



Chou and Foster (2014)

Physical picture: Chalker scaling, multifractality, and interactions

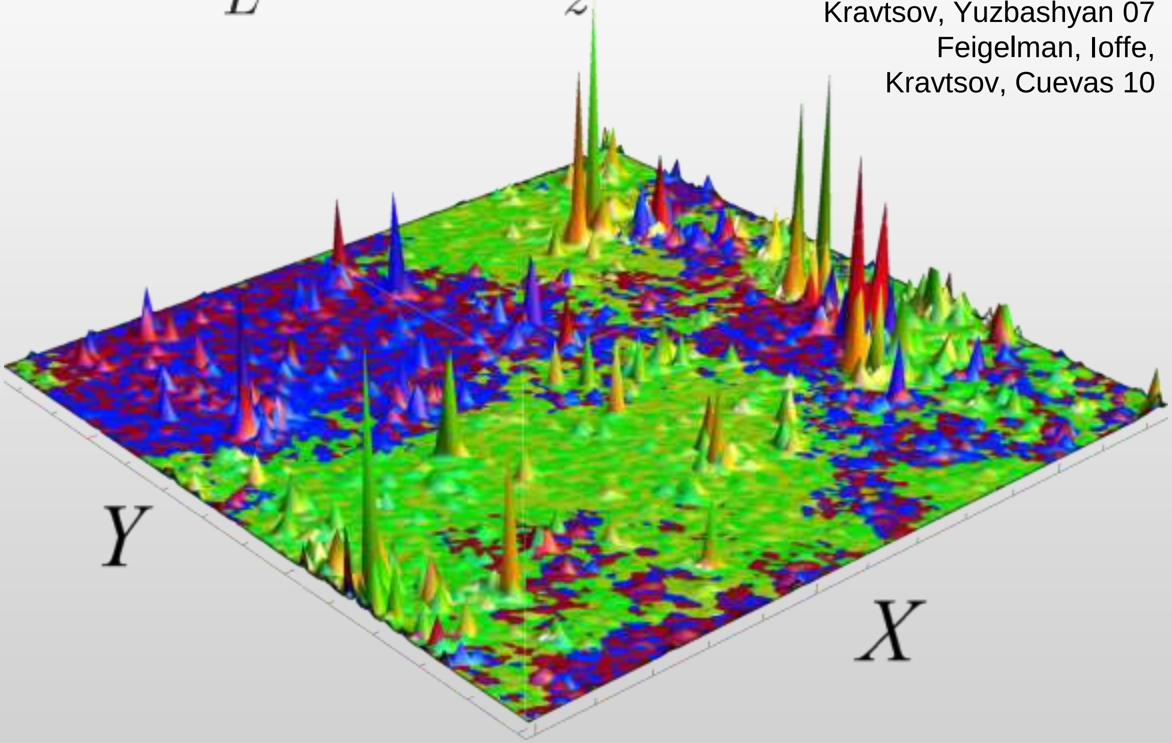
- **Chalker scaling: Overlapping peaks and valleys in multifractal eigenstates with nearby energies**

Chalker, Daniell 88
Chalker 90
Cuevas, Kravtsov 07

Feigelman, Ioffe, Kravtsov, Yuzbashyan 07
Feigelman, Ioffe, Kravtsov, Cuevas 10

$$\lim_{L\to\infty}\int d^2\mathbf{r}\,|\psi_0(\mathbf{r})|^2|\psi_\varepsilon(\mathbf{r})|^2 \sim \frac{\varepsilon^{-\mu}}{L^2},\quad \mu = \frac{2-\tau(2)}{z}$$

Probability peaks in *different* wavefunctions tend to cluster



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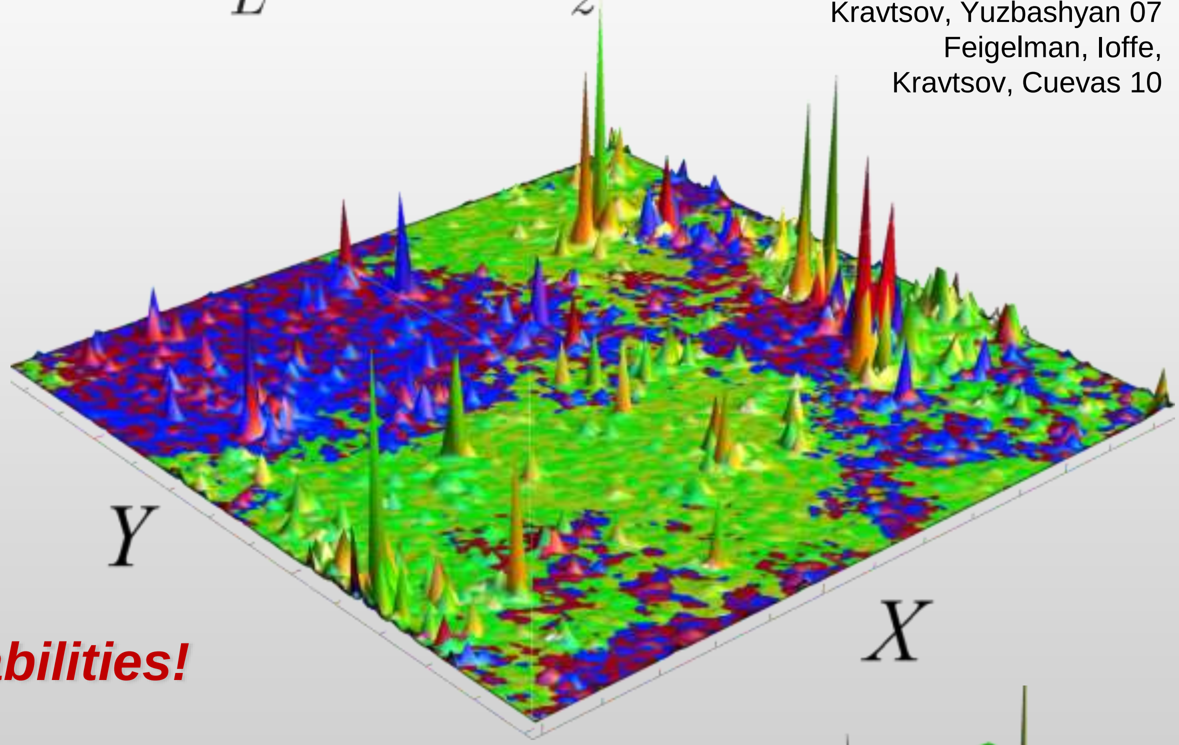
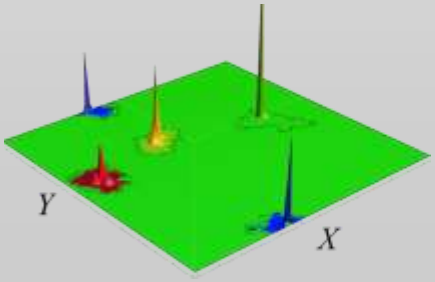


- **Why scaling theory of localization works**

- **Enhances interaction matrix elements *–instabilities!***

- **Anderson insulator: No overlap for nearby energies**

$$|\psi_\varepsilon(\mathbf{r})|^2|\psi_{\varepsilon'}(\mathbf{r})|^2 \sim 0,\quad 0 < |\varepsilon-\varepsilon'| \ll \delta_l$$

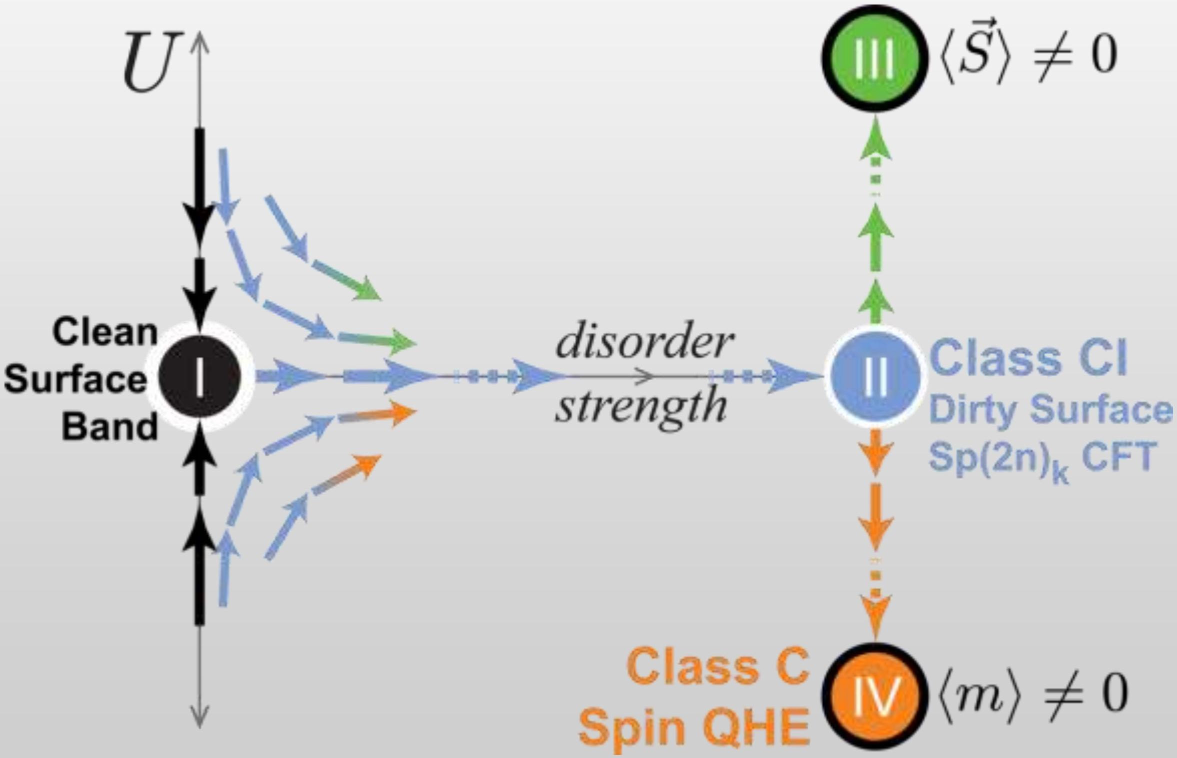


Weak disorder and interactions can sabotage topological protection!

- **<u>Result: Not always protected</u>. Even weak disorder and weak interactions can destroy some surface states**

Class CI Topological superconductors: Majorana surface fluid always unstable for any disorder, interactions, winding number

Foster, Yuzbashyan (2012)



Classes AIII, DIII: Stable surface states

Foster, Xie, Chou (2014)