Karlsruhe Institute of Technology

Depinning in disordered Josephson arrays and arrays of phase slip elements

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KIT & Landau Inst.

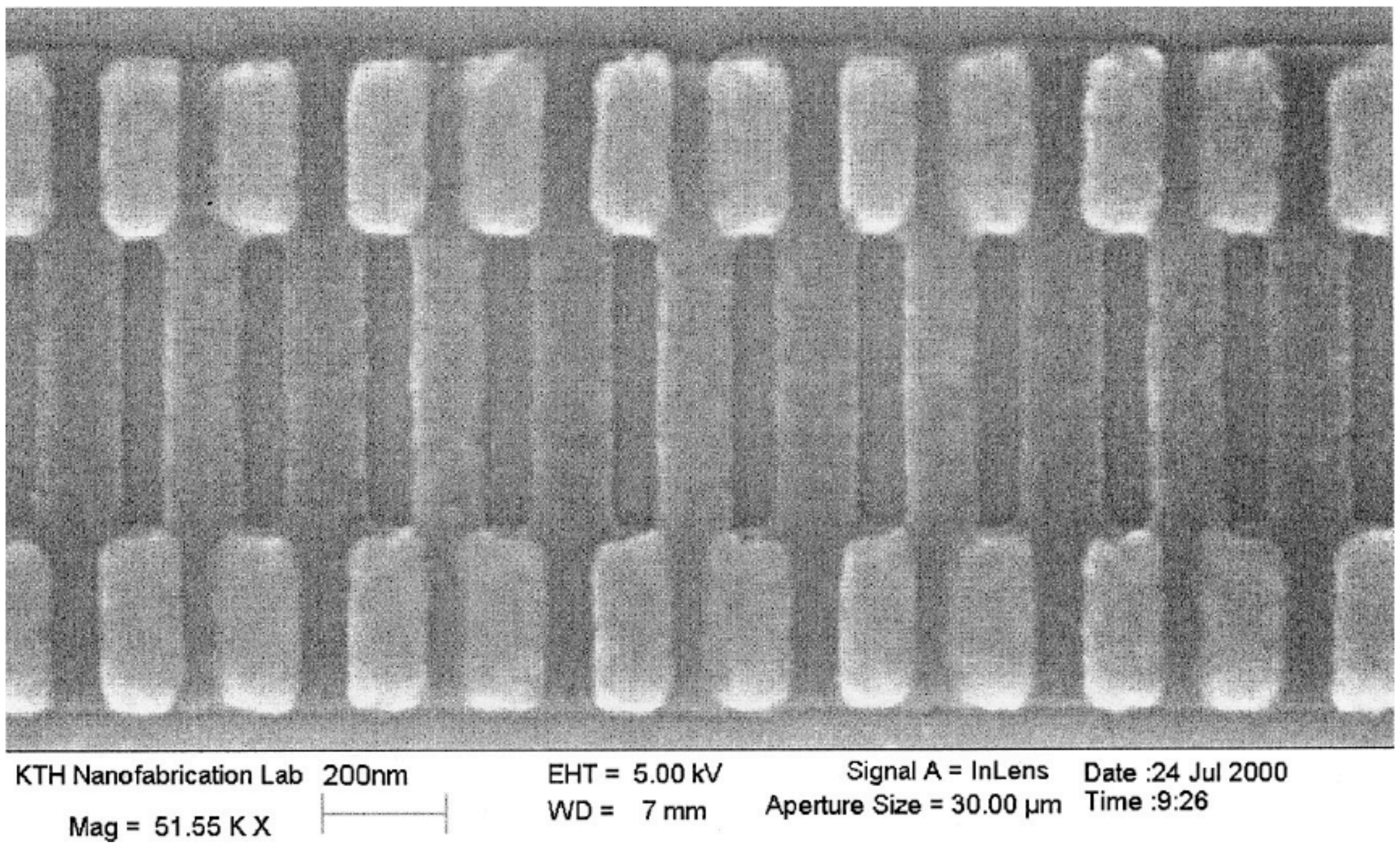
N. Vogt, R. Schäfer, H. Rotzinger, W. Cui, A. Fiebig, A.S. and A. V. Ustinov, cond-mat/1407.3353

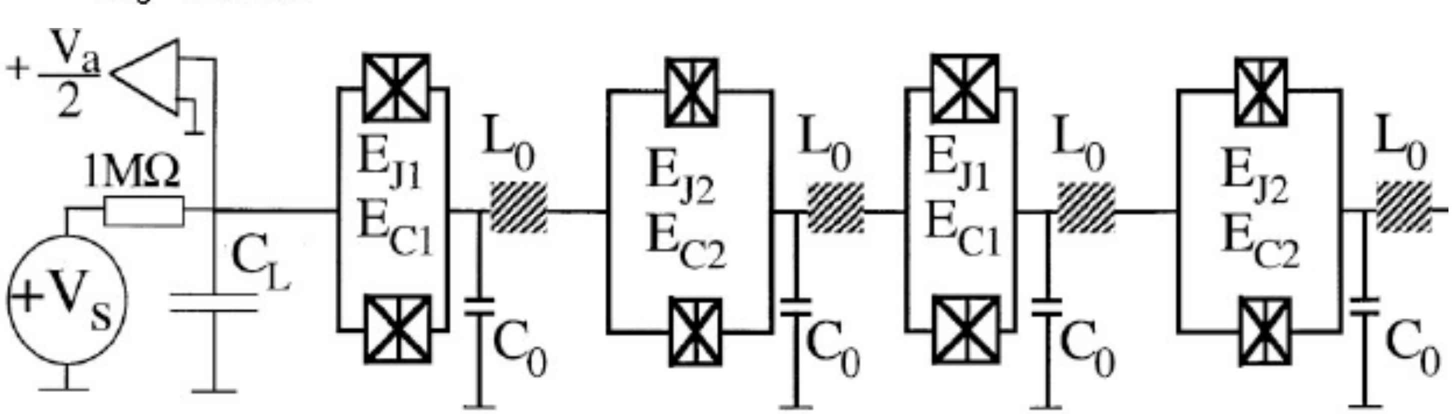
J E. Mooij, G. Schön, A.S., T. Fuse, C.J.P.M. Harmans, H. Rotzinger and A. H. Verbruggen, New J. Physics 17, 033006 (2015)

Experimental motivation

Haviland, Delsing, PRB 1996 **Ågren, Andersson, Haviland, JLTP 2000,2001**

R. Schäfer et al., arXiv:1310.4295





30-5000 junctions in array

$$E_J(\Phi) = 2E_{J,0}\cos\frac{\pi\Phi}{\Phi_0}$$

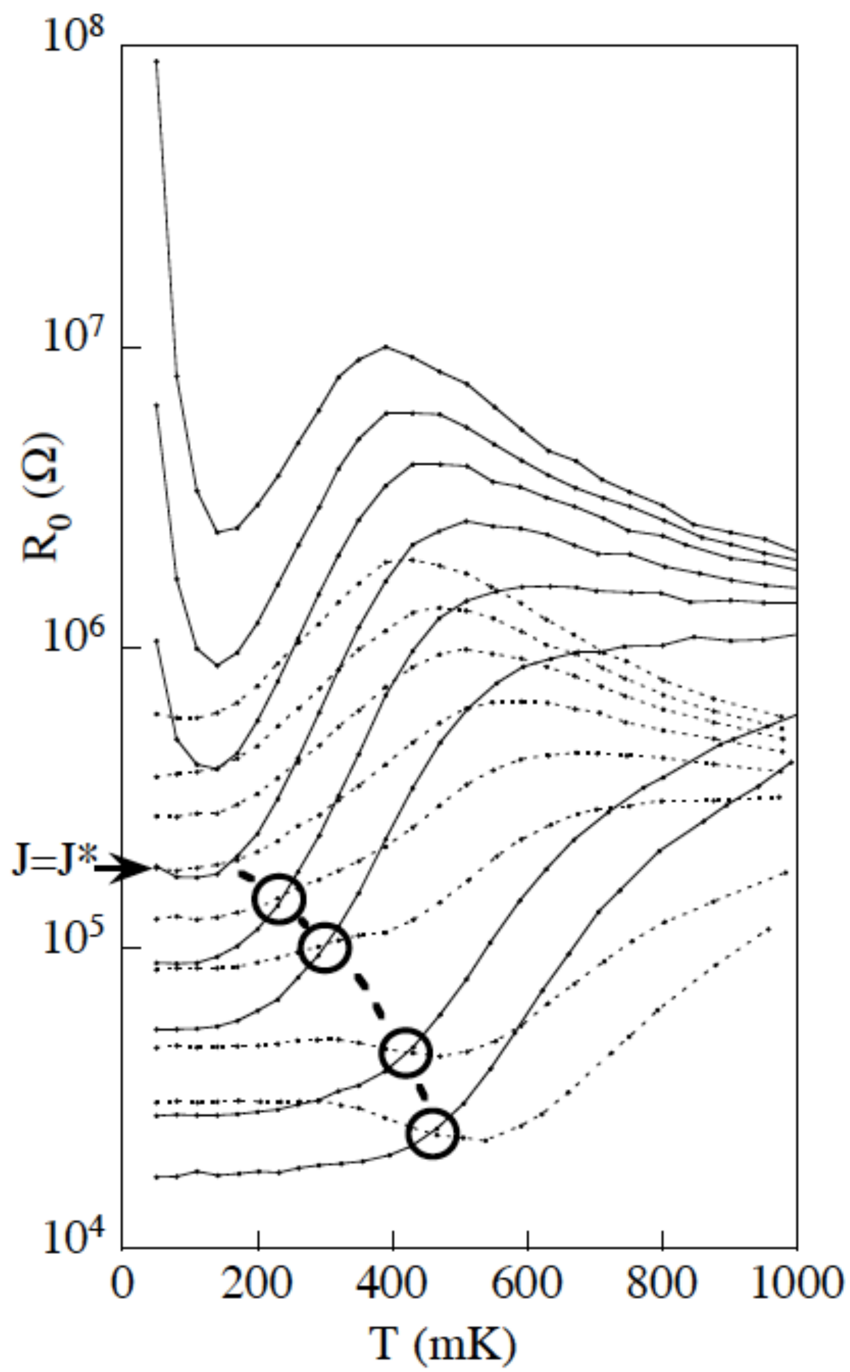
$$E_{J,0} \sim E_C \equiv \frac{(2e)^2}{2C}$$

$$E_{C_0} \equiv \frac{(2e)^2}{2C_0} \gg E_C$$

Poor screening

Experimental motivation

- **(Super)conductor - insulator (Coulomb blockade) transition**
- **Linear response**

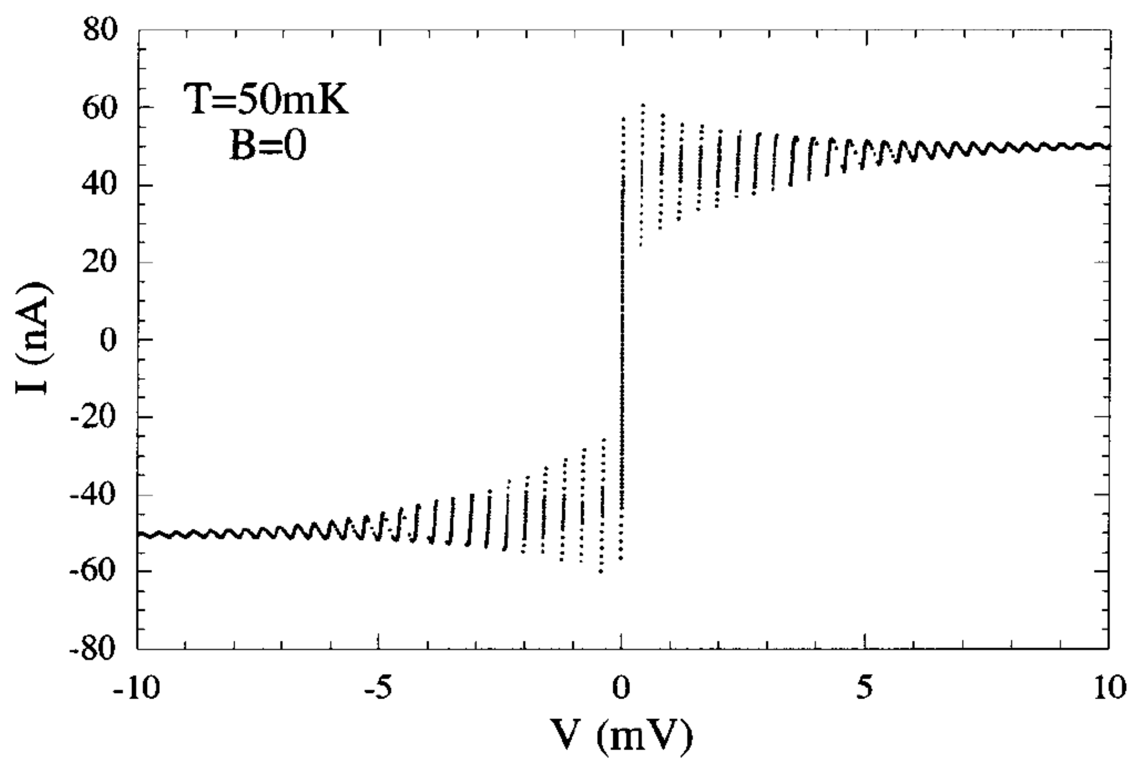


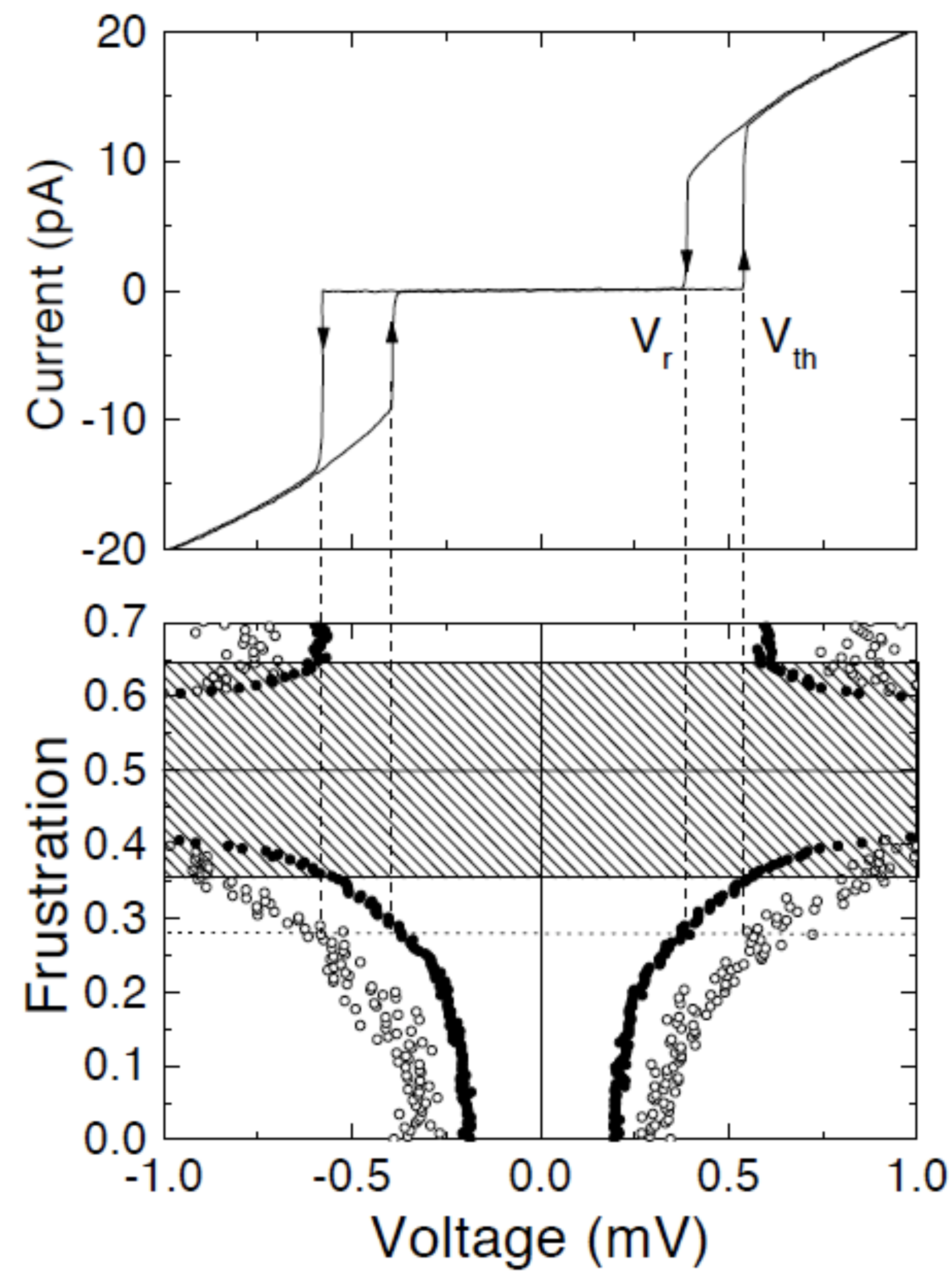
Chow, Delsing , Haviland,, PRL 1997

Experimental motivation

- (Super)conductor - insulator transition: non-linear response

$E_J(\Phi) > E_C$ $\qquad$ $E_J(\Phi) \lesssim E_C$





- Flux dependent Coulomb blockade
- Hysteresis

Ågren, Andersson, Haviland, JLTP 2000,2001

Experimental motivation

R. Schäfer, H. Rotzinger, A. Ustinov (private communication, 2010-2012)
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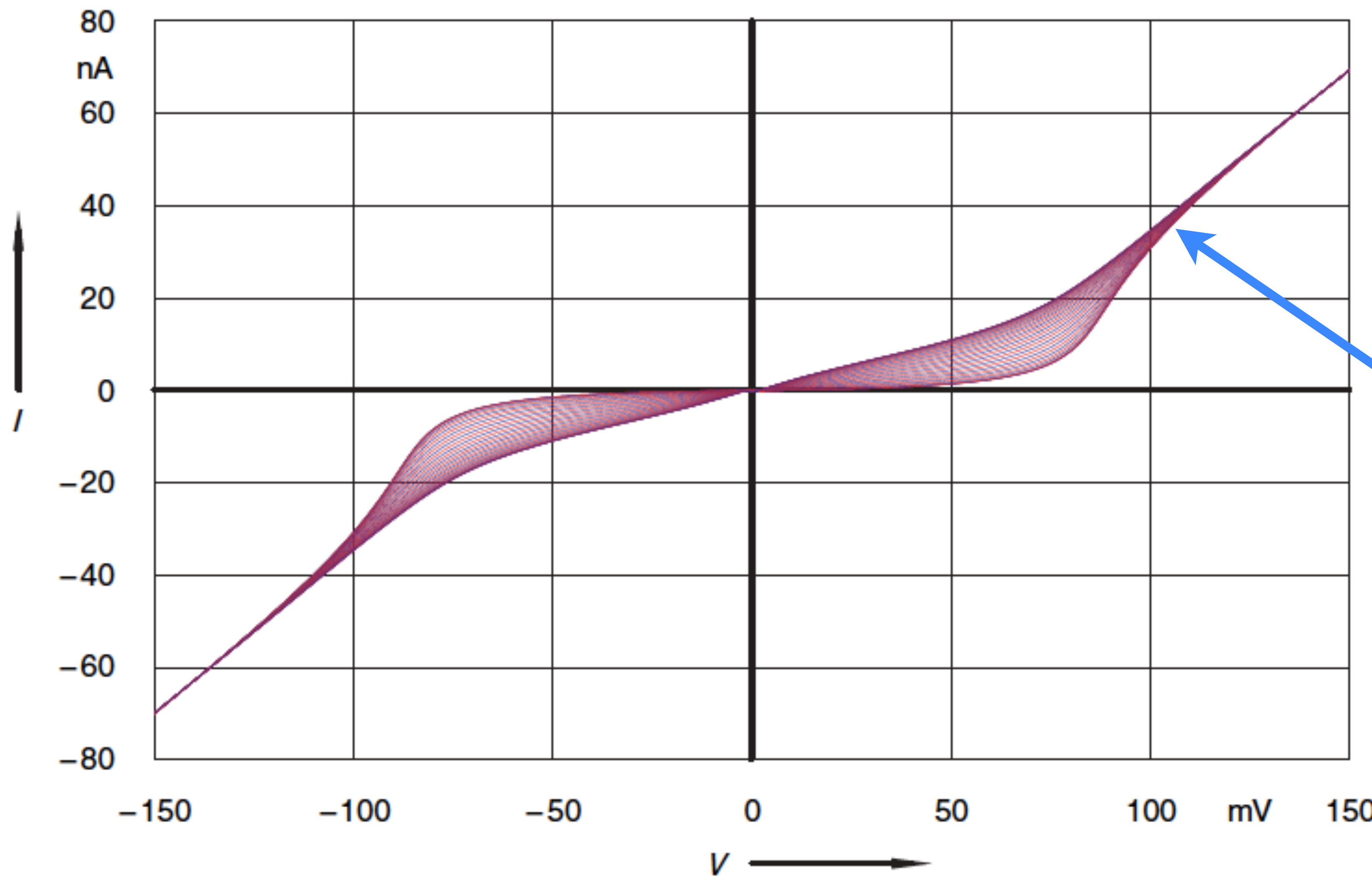


$$\frac{eV}{N} \sim 2\Delta$$

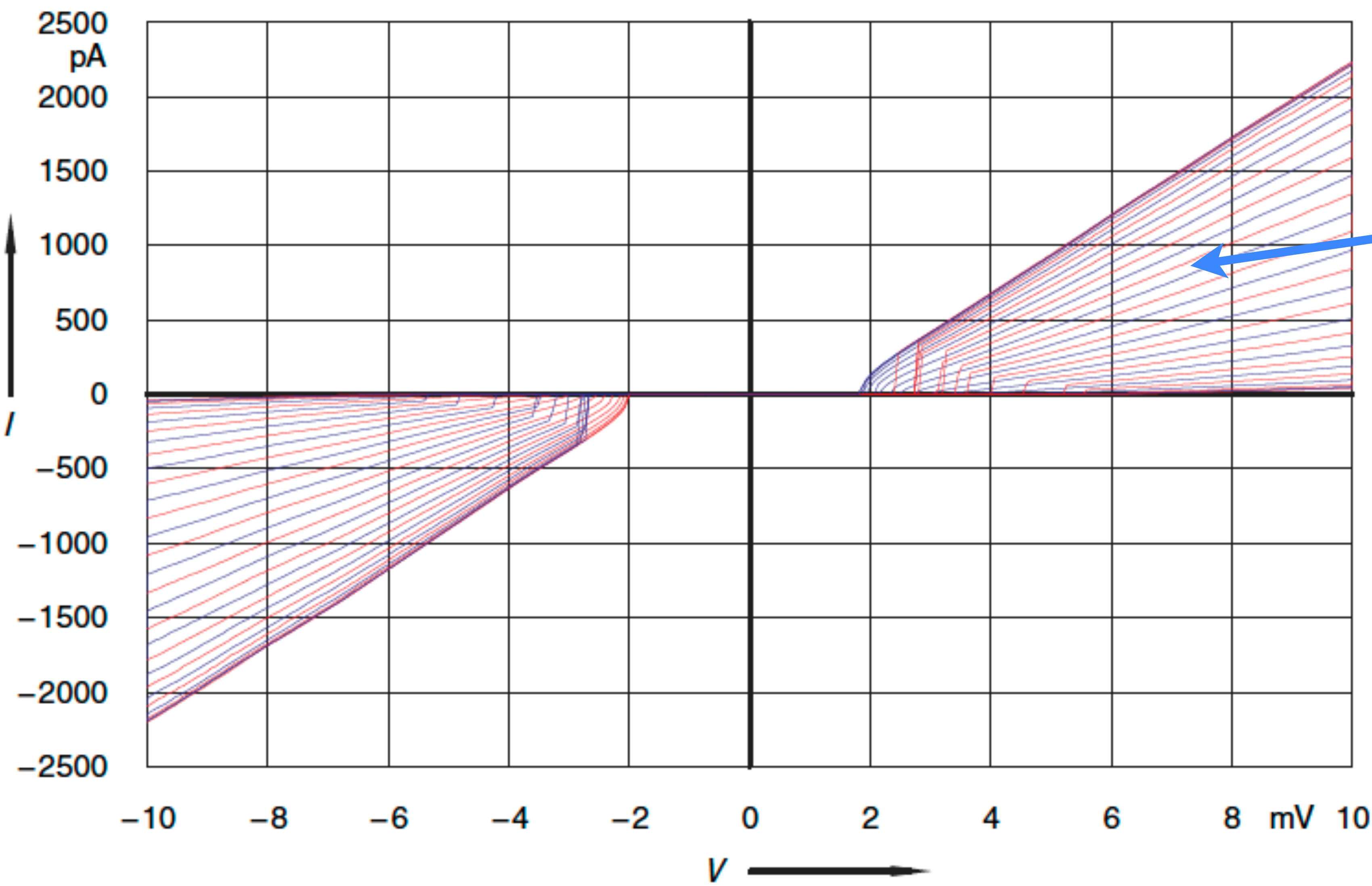
$$N \sim 250$$

- **Flux dependent dissipative transport of Cooper pairs at high voltages**

Experimental motivation

R. Schäfer, H. Rotzinger, A. Ustinov (private communication, 2010-2012)

R. Schäfer et al., arXiv:1310.4295



$$\frac{dI}{dV} \propto [E_{\mathrm{J}}(\Phi)]^2$$

$$\frac{eV}{N} \ll 2\Delta$$

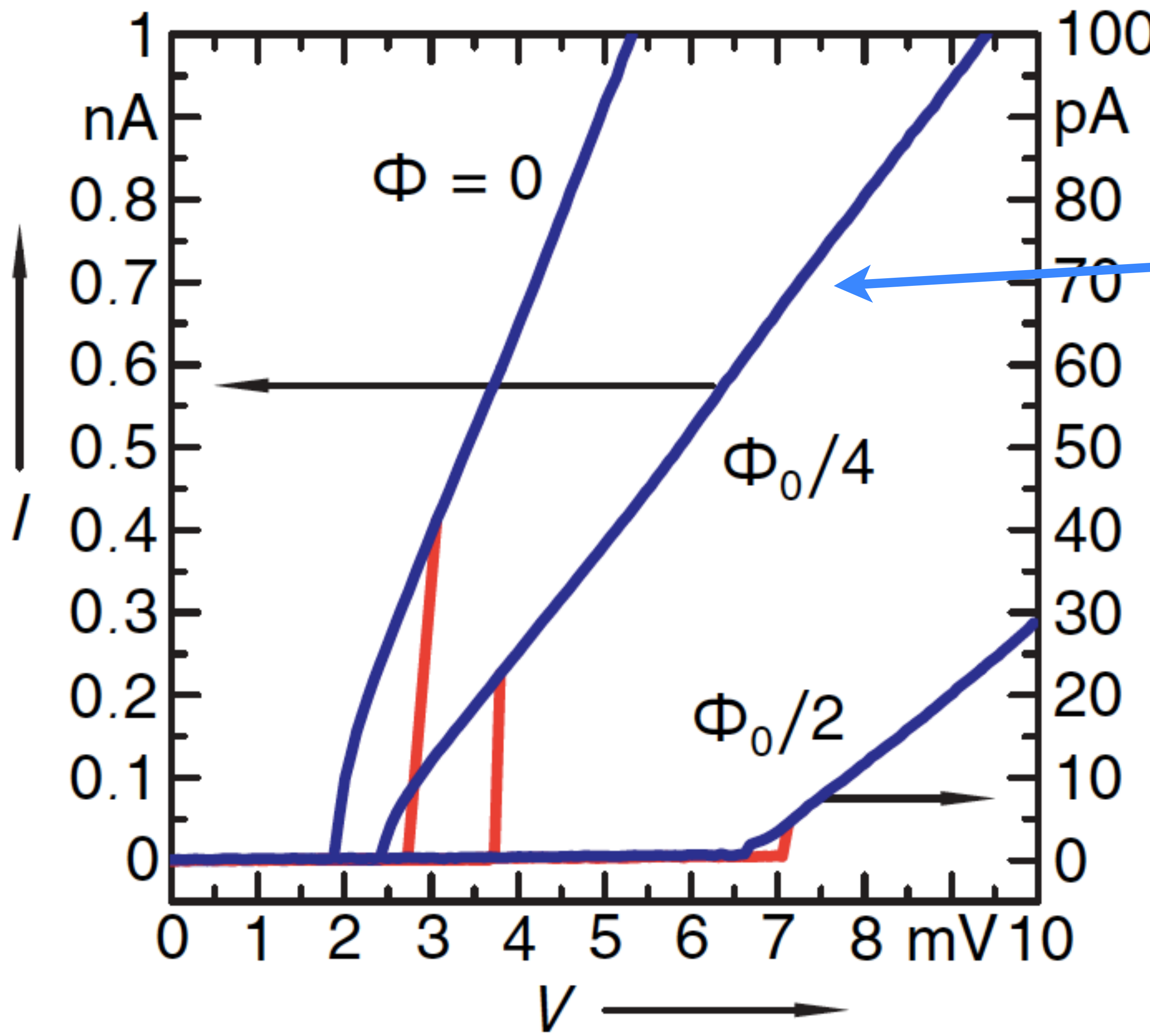
No quasiparticles ?

- **Flux dependent dissipative transport of Cooper pairs at high voltages**
- **Flux dependent Coulomb blockade**
- **Hysteresis**

Experimental motivation

R. Schäfer et al., arXiv:1310.4295

Flux-dependent switching voltage



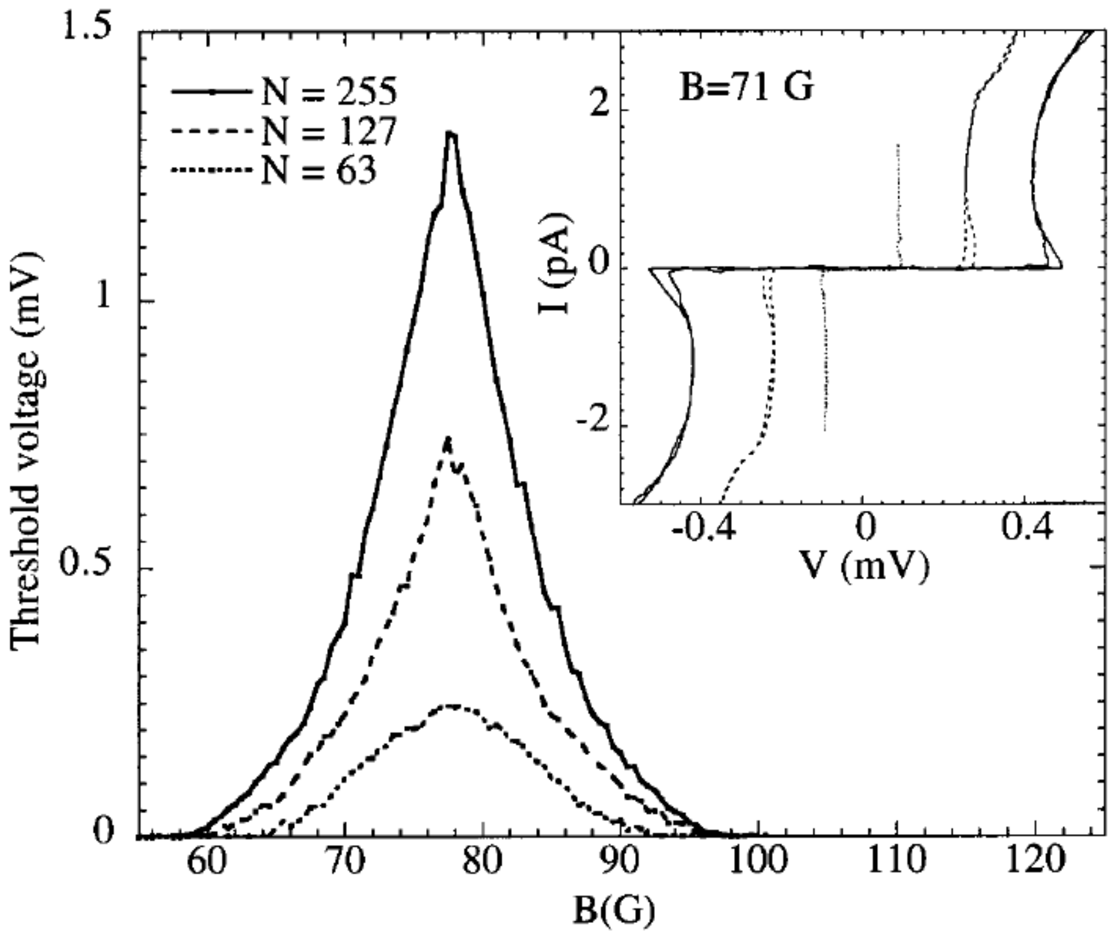
$$\frac{dI}{dV} \propto [E_{\mathrm{J}}(\Phi)]^2$$

$$\frac{eV}{N} \ll 2\Delta$$

No quasiparticles ?

Experimental motivation

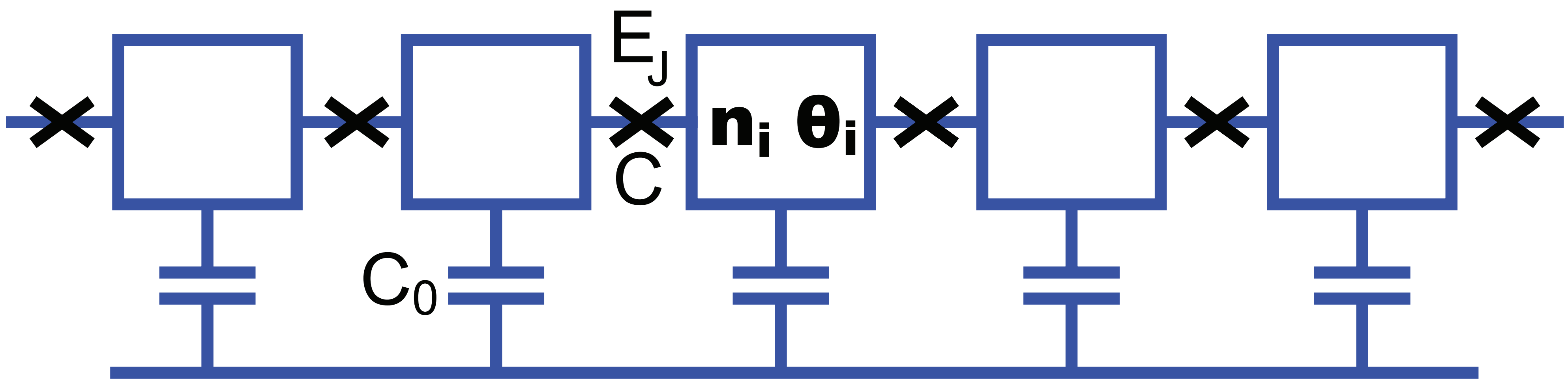
Ågren, Andersson, Haviland, JLTP 2000,2001



Threshold voltage scales linearly with array length
No effect of overall gate voltage

Strong disorder

The model (no disorder)



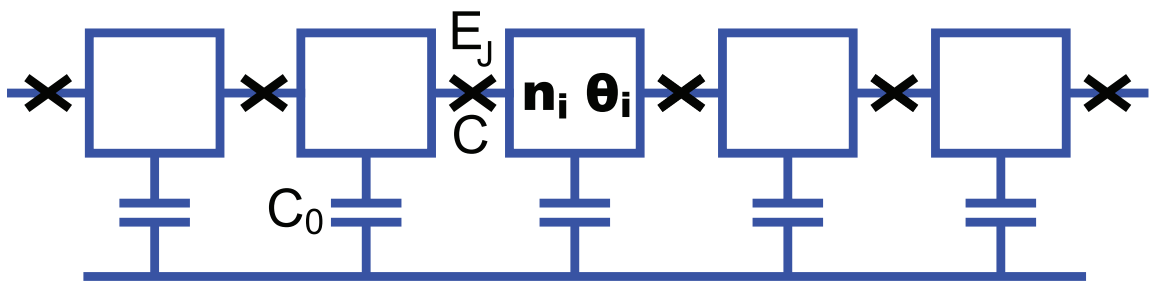
$$H = \frac{1}{2}\sum_{i,j} U(i-j)\, n_i n_j - \sum_i E_{\mathrm{J}} \cos(\theta_{i+1} - \theta_i)$$

$$[n_j, e^{i\theta_{j'}}] = e^{i\theta_j}\,\delta_{j,j'} \qquad U(i-j) = (2e)^2(\hat{C}^{-1})_{i,j}$$

$$(\hat{C})_{i,j} = C_0\delta_{i,j} + C\,[2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}]$$

Review: R. Fazio and H. van der Zant, Phys. Rep. 355, 235 (2001)

Charging energy



Charging energy $E_C \equiv \frac{(2e)^2}{2C}$

Screening length $\Lambda \equiv \sqrt{\frac{C}{C_0}}$

$$E_{C_0} \equiv \frac{(2e)^2}{2C_0} \gg E_C$$

$$H_C = \frac{1}{2}\sum_{i,j} U(i-j)\, n_i n_j$$

$$\Lambda \equiv \sqrt{\frac{C}{C_0}} \gg 1 \quad \Rightarrow \quad U(i-j) \approx \Lambda E_C e^{-\frac{|i-j|}{\Lambda}}$$

BKT quantum phase transition

$$K = \frac{\pi}{\Lambda}\sqrt{\frac{E_J}{2E_C}}$$

$K > 2$ Insulator

$K < 2$ Superconductor

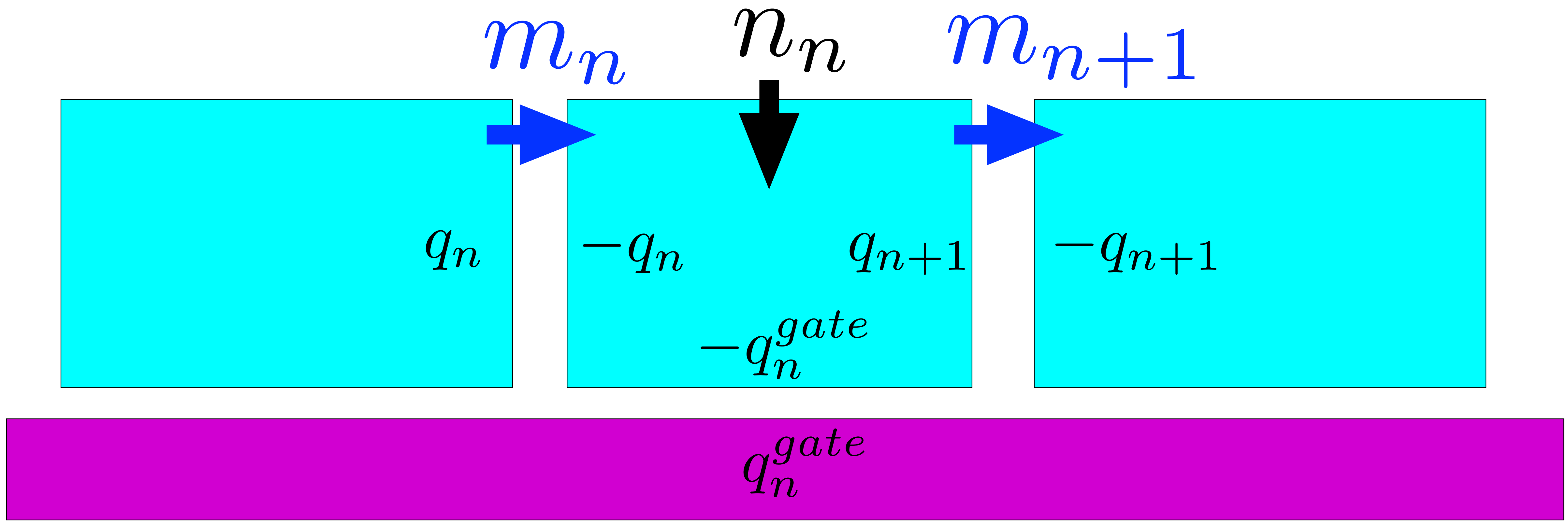
S. E. Korshunov, Sov. Phys. JETP 68, 609 (1989)
M.Y. Choi et al., Phys. Rev. B **48**, 15 920 (1993)
G. Rastelli, I. M. Pop, and F. W. J. Hekking, Phys. Rev. B 87, 174513 (2013)

$$\Lambda \to \infty \Rightarrow \text{insulator for } E_J > E_C$$

In experiment: “transition” mostly at $E_J \sim E_C$

Quasi-charge description

Various charge variables



Charge conservation $2en_n = 2e(m_n - m_{n+1}) = q_{n+1} - q_n - q_n^{gate}$

$$Q_n = const. + \sum_{m<n} q_m^{\mathrm{gate}} = q_n + 2em_n$$

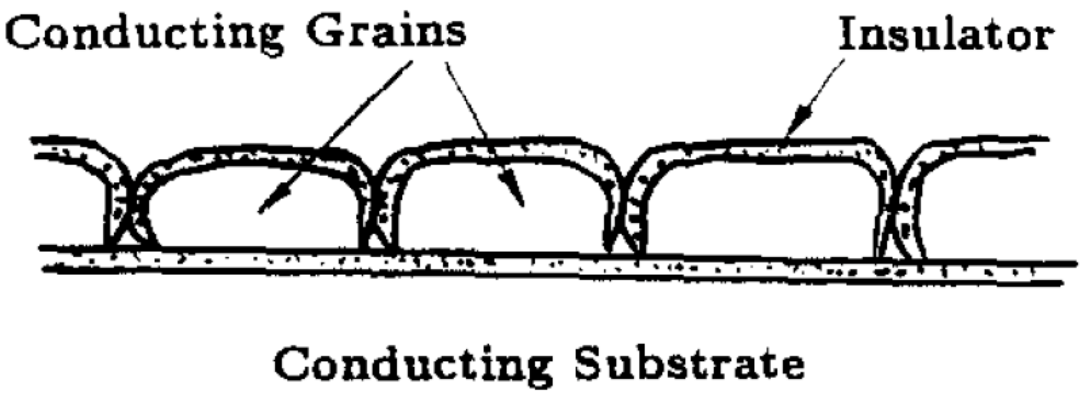
Displacement charge on junction n

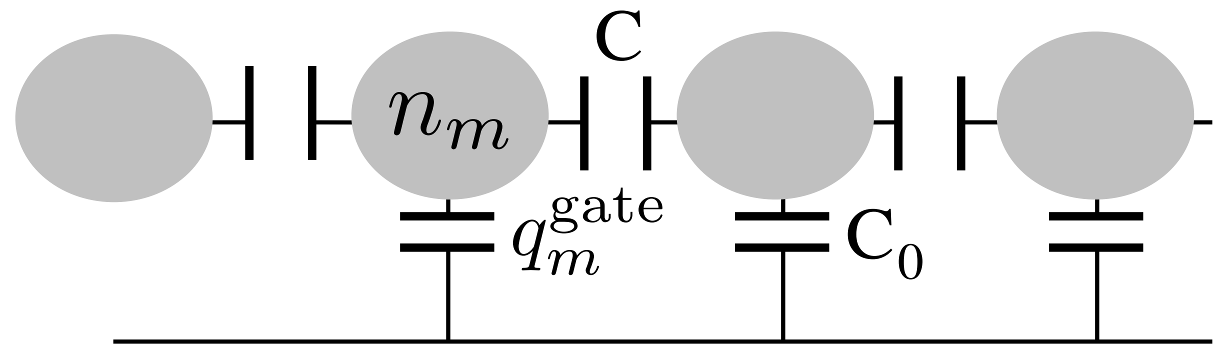
$$q_n = Q_n - 2em_n$$

Charge that has arrived at junction n

$$Q_n(t) = \int^t I_n(t')dt' \qquad const. = 2em_{-\infty}$$

Idea of charge solitons in arrays of tunnel junctions





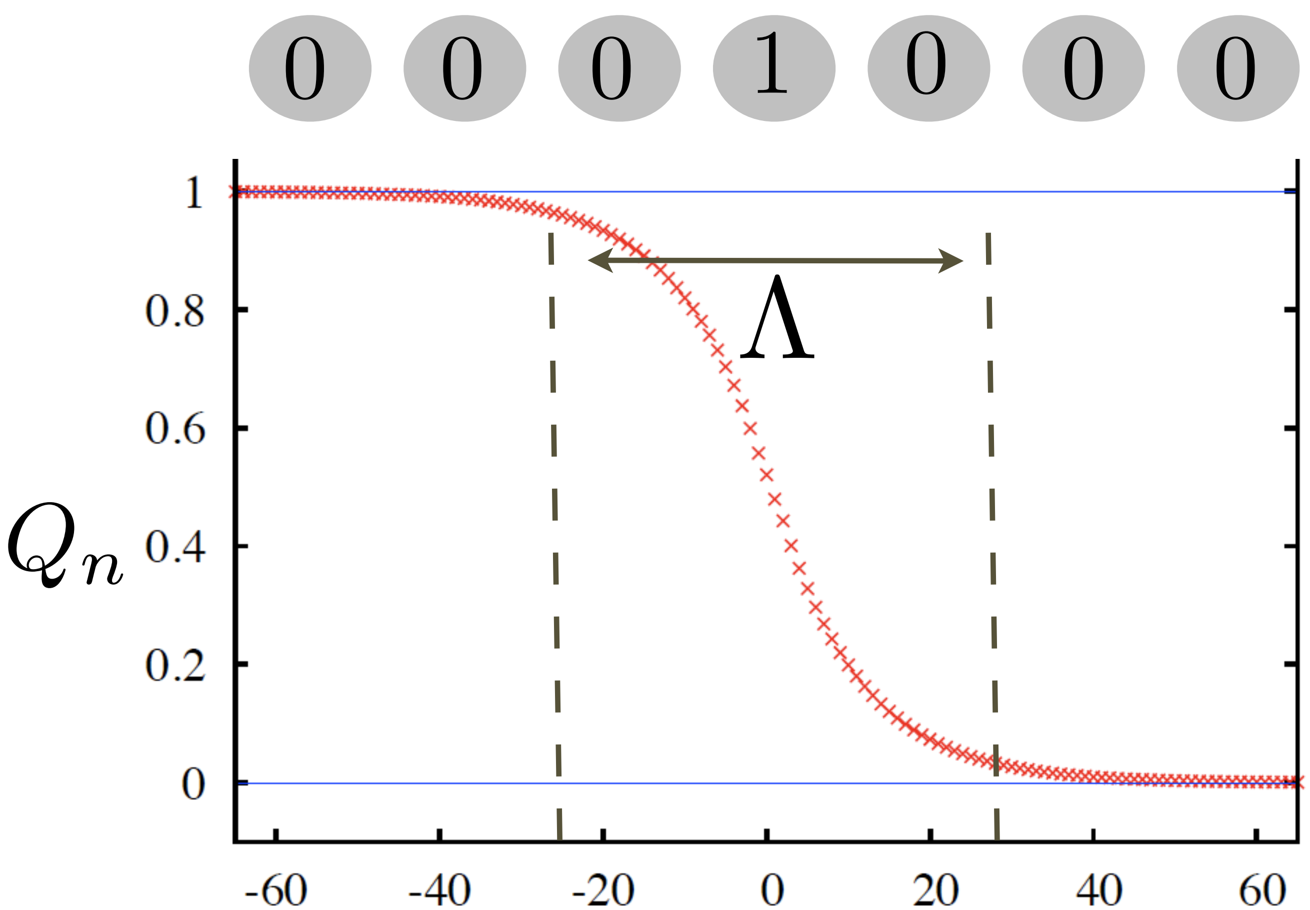
Ben-Jacob, Mullen, Amman (1989)

Likharev et al. (1989)

$$V_n = \frac{Q_n - 2em_n}{C} = \frac{Q_{n+1} + Q_{n-1} - 2Q_n}{C_0}$$

Charge that has arrived at junction n

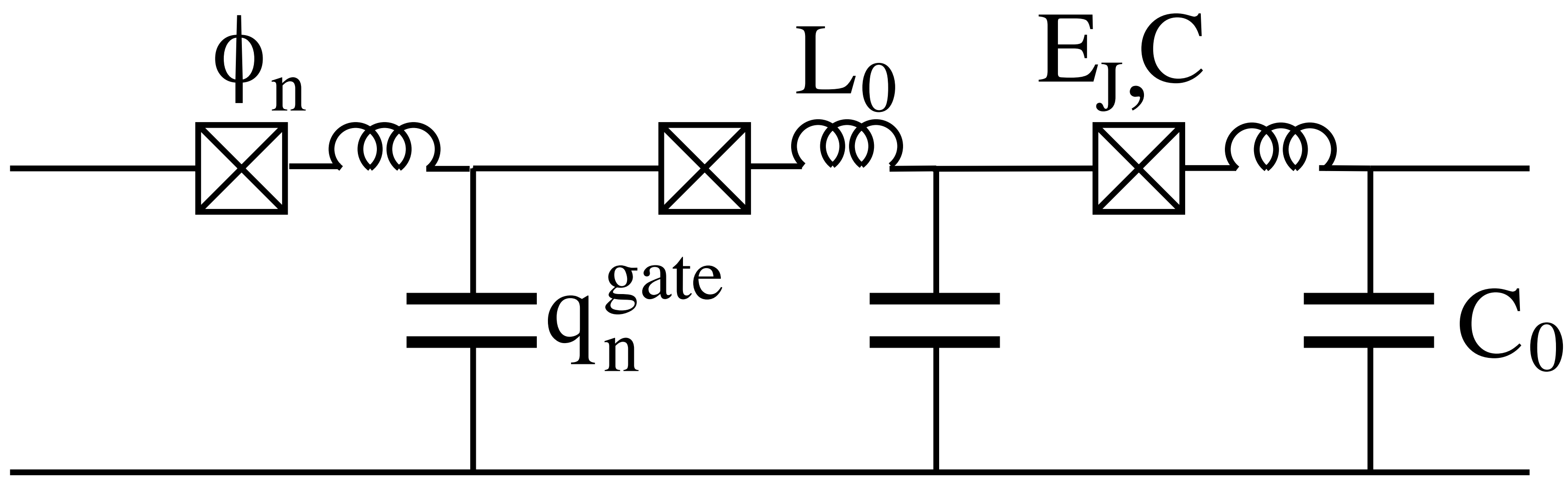
$$Q_n = const. + \sum_{m<n} q_m^{\mathrm{gate}} = \int^t I_n(t')dt'$$



$$\Lambda \equiv \sqrt{\frac{C}{C_0}}$$

Screening length

Quasi-charge description



$$H=\sum_n\left[\frac{(Q_n-2em_n)^2}{2C}-E_{\mathrm{J}}\cos\phi_n+\frac{(Q_n-Q_{n-1})^2}{2C_0}+\frac{\Phi_n^2}{2L_0}\right]$$

Continuous charge that has flown into junction n

$$Q_n=const.+\sum_{m<n}q_m^{\mathrm{gate}}$$

$$[\Phi_n,Q_{n'}]=i\hbar\delta_{n,n'}$$

Quantized charge that has tunneled through junction n

$$2em_n$$

$$[m_n,e^{i\phi_{n'}}]=e^{i\phi_n}\delta_{n,n'}$$

Limit of large (kinetic) inductance

Hermon, Ben-Jacob, Schön, PRB 96

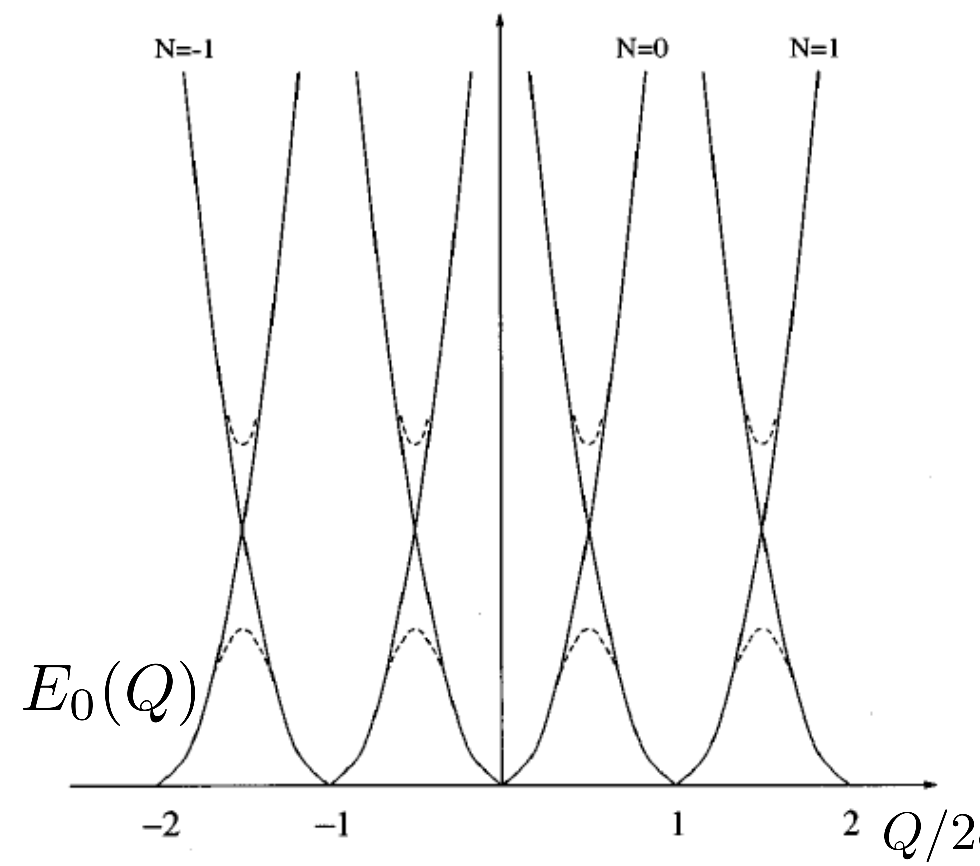
Gurarie, Tsvelik, JLTP 03

$$H = \sum_n \left[\frac{(Q_n - 2em_n)^2}{2C} - E_\mathrm{J} \cos\phi_n + \frac{(Q_n - Q_{n-1})^2}{2C_0} + \frac{\Phi_n^2}{2L_0} \right]$$

$[\Phi_n, Q_{n'}] = i\hbar\delta_{n,n'}$ slow variables

$[m_n, e^{i\phi_{n'}}] = e^{i\phi_n}\delta_{n,n'}$ fast variables

Josephson junction energy bands



Born-Oppenheimer approximation

$$H = \sum_n \left[E_0(Q_n) + \frac{(Q_n - Q_{n-1})^2}{2C_0} + \frac{\Phi_n^2}{2L_0} \right]$$

“sine”-Gordon equation of motion

$$L_0 \ddot{Q}_n + \frac{2Q_n - Q_{n+1} - Q_{n-1}}{C_0} + \frac{\partial E_0}{\partial Q_n} = 0$$

Bloch inductance

Zorin PRL 2006

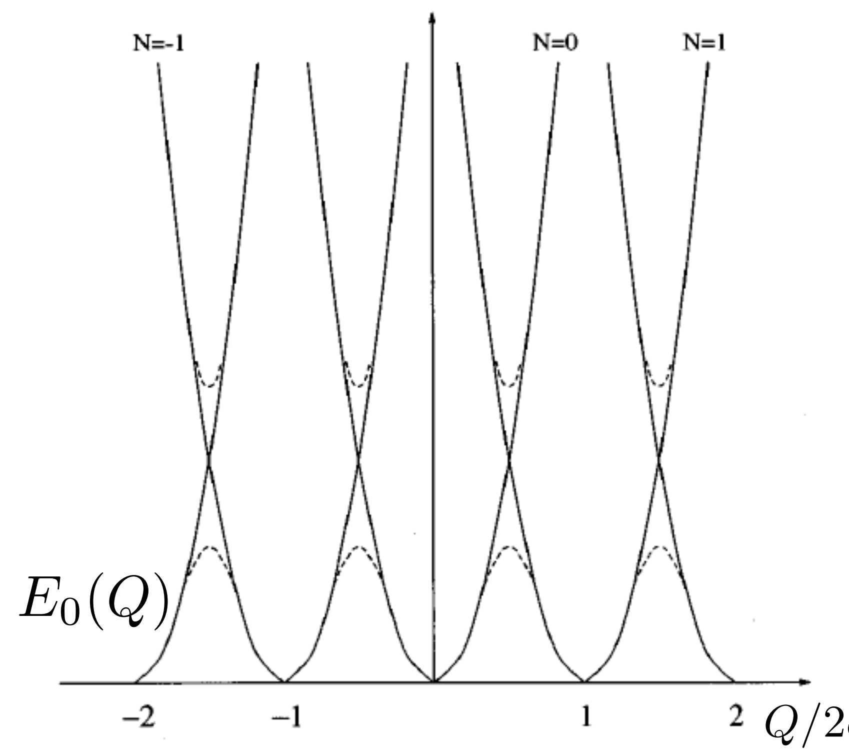
Single current biased JJ

$$H(Q(t)) = \frac{(2em - Q(t))^2}{2C} - E_\mathrm{J}\cos\phi$$

Voltage (adiabatic case)

$$V = \left\langle \frac{Q - 2em}{C} \right\rangle = \left\langle \frac{\partial H}{\partial Q} \right\rangle$$

$$= \frac{\partial E_0}{\partial Q} + L_B(Q)\ddot{Q} + \frac{1}{2}\left[\partial_Q L_B(Q)\right]\dot{Q}^2$$



Josephson junction energy bands

Euler - Lagrange Eq. for

$$L_\mathrm{eff}(Q,\dot{Q}) = \frac{L_B(Q)\dot{Q}^2}{2} - E_0(Q) - VQ$$

$$L_J = \frac{1}{E_J}\left(\frac{\Phi_0}{2\pi}\right)^2$$

$$L_B(Q) \approx L_J \text{ for } E_J \geq E_C$$

J. Homfeld, I. Protopopov, S. Rachel, and A. S., Phys. Rev. B 83, 064517 (2011)

“sine-Gordon” Lagrangian

$$\mathcal{L} = \sum_n \left[\frac{1}{2} \left[L_0 + L_B(Q_n) \right] \dot{Q}_n^2 - \frac{(Q_n - Q_{n-1})^2}{2C_0} - E_0(Q_n) \right]$$

$$L_B(Q) \approx L_J \text{ for } E_J \geq E_C$$

$$L_J = \frac{1}{E_J} \left(\frac{\Phi_0}{2\pi} \right)^2$$

Bloch Inductance

$$L_0 \rightarrow 0$$

No longer needed

J. Homfeld, I. Protopopov, S. Rachel, and A. S., Phys. Rev. B 83, 064517 (2011)

Luttinger Lagrangian for $E_J \gg E_C$

$$\mathcal{L} = \frac{1}{2\pi K} \sum_n \left[\frac{\dot{q}_n^2}{v} - v \left(q_n - q_{n+1} \right)^2 \right] + \sum_n E_S \cos\left(2q_n\right)$$

$$q_n = \pi Q_n / (2e)$$

$$v \equiv \frac{1}{\sqrt{L_J C_0}}$$

$$K = \pi \sqrt{\frac{E_J}{2E_{C0}}} = \frac{\pi}{\Lambda} \sqrt{\frac{E_J}{2E_C}} \ll 1$$

E_S phase slip amplitude - relevant perturbation

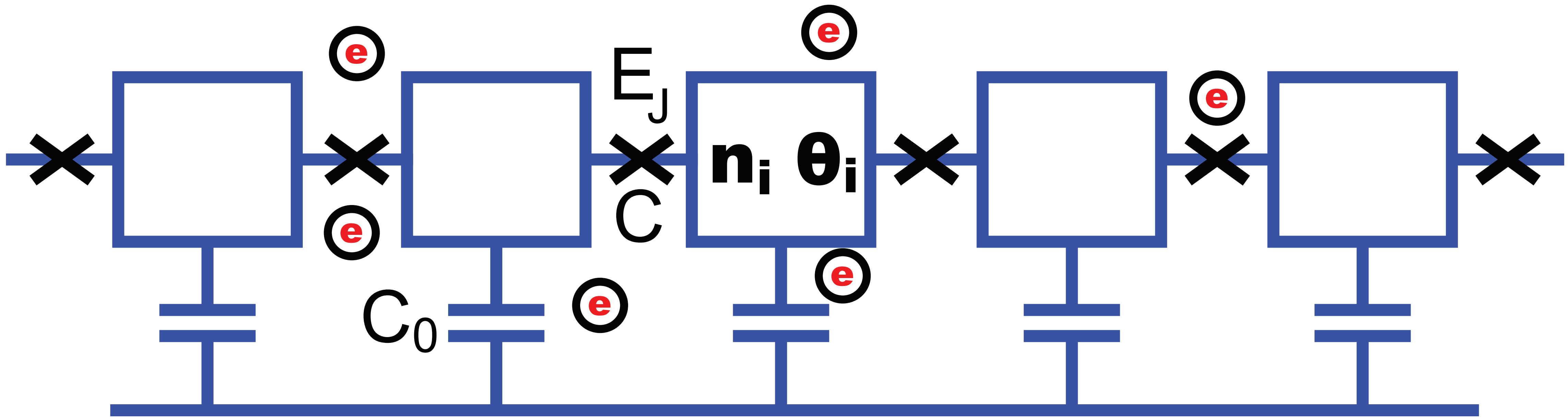
Classical dynamics of $Q_n(t)$

J. Homfeld, I. Protopopov, S. Rachel, and A. S., Phys. Rev. B 83, 064517 (2011)

Charge disorder: pinning

Gurarie, Tsvelik, JLTP 03

Offset charges



$$H = \frac{1}{2}\sum_{i,j} U_{i,j}\,(n_i + \delta q_i)(n_j + \delta q_j) - \sum_i E_J \cos(\theta_{i+1} - \theta_i)$$

Offset charges

$$H = \frac{1}{2}\sum_{i,j} U_{i,j}\,(n_i + \delta q_i)(n_j + \delta q_j) - \sum_i E_J \cos(\theta_{i+1} - \theta_i)$$

quasi-charge description

$$\mathcal{L} = \sum_n \left[\frac{1}{2} L_B (Q_n + F_n)\dot{Q}_n^2 - \frac{(Q_n - Q_{n-1})^2}{2C_0} - E_0(Q_n + F_n)\right]$$

$$F_n = 2e \sum_{i=-\infty}^{n} \delta q_i$$

Strong disorder:

$$P(\delta q) = const.\ \text{for}\ \delta q \in [-1/2, 1/2]$$

$\Downarrow$ mod $(2e)$

$$\langle F_n F_m \rangle \sim \delta_{n,m}$$

Charging energy

$$H_c = \sum_n \left[\frac{(Q_n - Q_{n+1})^2}{2C_0} + U\left[Q_n + F_n\right] - E\, Q_n \right] \ ,$$

$$H_c = \int dx \left[\frac{(\partial_x Q(x))^2}{2C_0} + U\left[Q(x) + F(x)\right] - E\, Q(x) \right]$$

$U[Q] \equiv E_0(Q)$ Lowest Bloch band

Larkin length

$$H_c = \sum_n \left[\frac{(Q_n - Q_{n+1})^2}{2C_0} + U\left[Q_n + F_n\right] \right]$$

$$\langle [Q_n - Q_m]^2 \rangle \sim e^2 \left(\frac{|n-m|}{N_L} \right)^{(4-D)}$$

Compare elastic and pinning energy

$$N_L \times \frac{[e/N_L]^2}{2C_0} \sim \sqrt{N_L}\,\Delta U$$

$$\Delta U \sim (U_{max} - U_{min})$$

$$N_L \sim \left(\frac{E_{C0}}{\Delta U} \right)^{2/3} \sim \Lambda^{4/3} \tilde{R}^{-1/3}$$

$$\tilde{R} \equiv \frac{(\Delta U)^2}{E_C^2}$$

A. I. Larkin, Sov. Phys. JETP 31, 784 (1970)

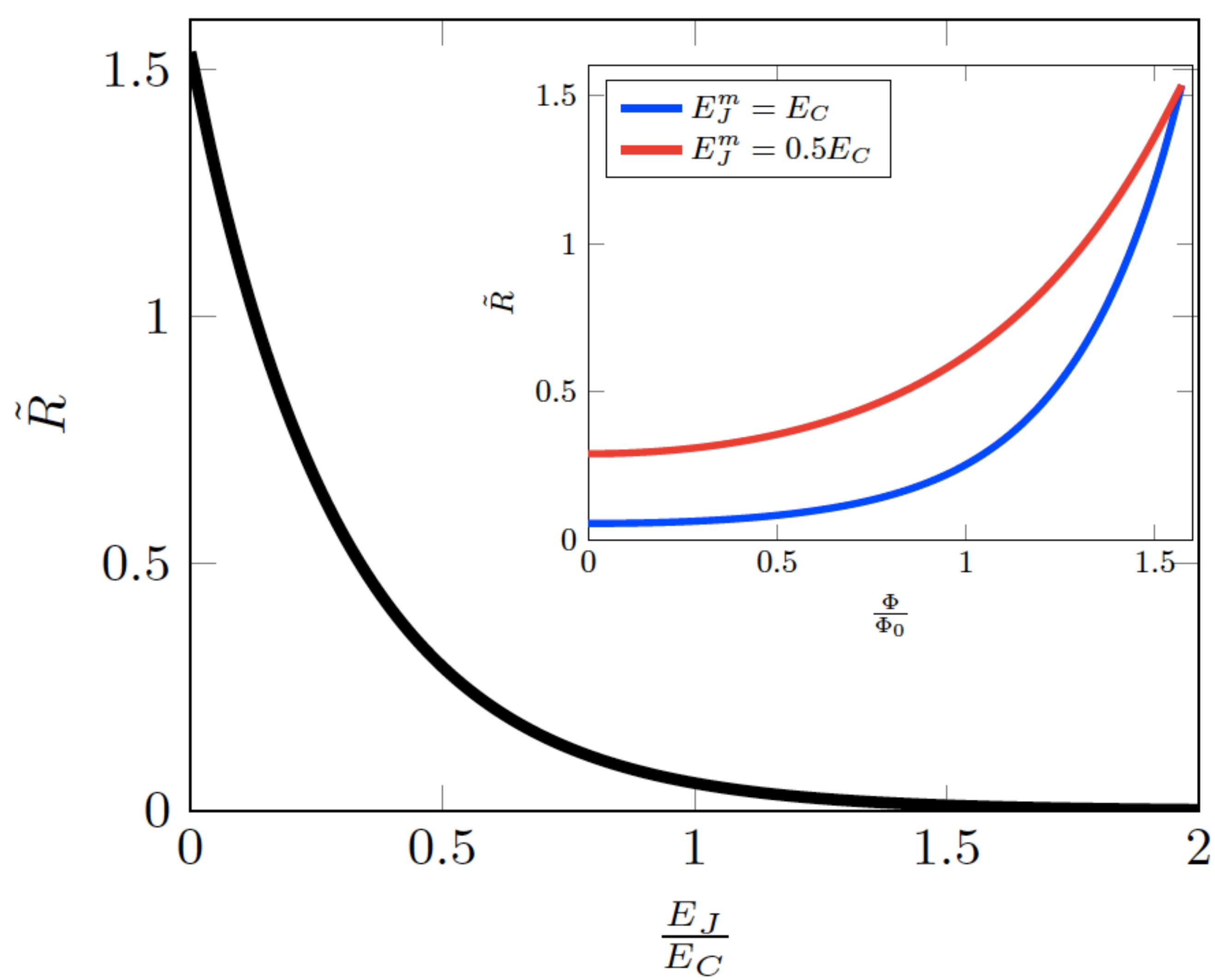
Y. Imry and S.-K. Ma, Phys. Rev. Lett. **35**, 1399 (1975)

H. Fukuyama and P. A. Lee, Phys. Rev. B **17**, 535 (1978)

Depinning

Depinning field $E_p \approx \dfrac{e}{C_0 N_L^2}$

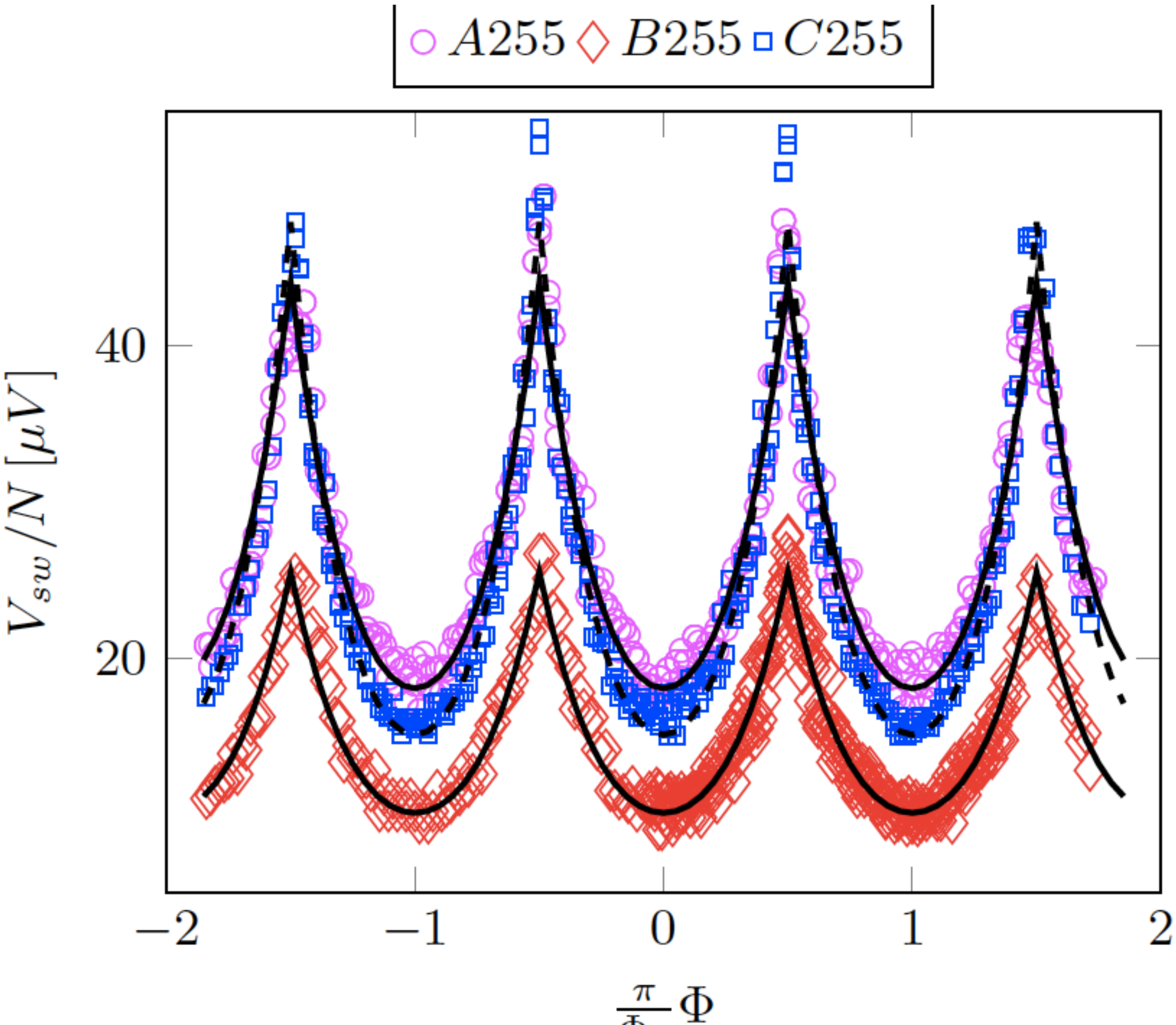
$$V_{sw} \approx \frac{NE_C}{2e}\,\Lambda^{-\frac{2}{3}}\,\tilde{R}^{2/3}$$



$$\tilde{R} \equiv \frac{(\Delta U)^2}{E_C^2}$$

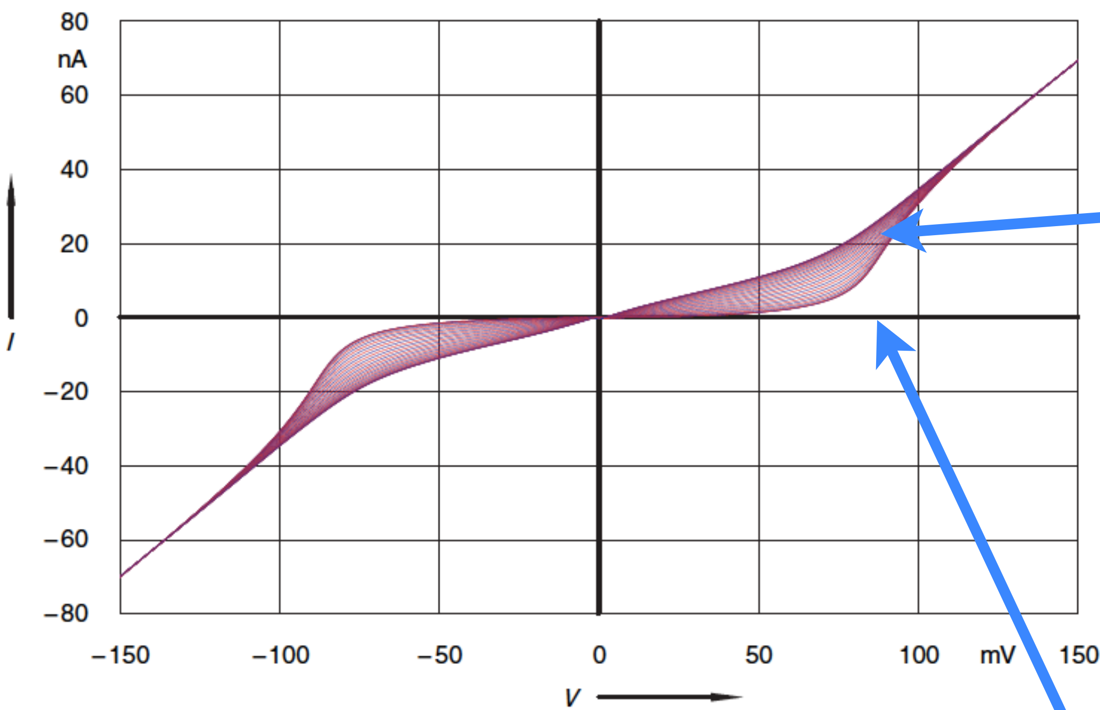
Disorder

$$\frac{V_{sw}}{N} \approx \frac{E_C}{2e} \Lambda^{-\frac{2}{3}} \tilde{R}^{2/3} \left[E_J(\Phi)/E_C\right]$$



Transport after depinning

Monte-Carlo simulations with quasi-particles and for $E_J \ll E_C$
J. H. Cole, J. Lepp!akangas, and M. Marthaler, New J. of Physics 16, 063019 (2014)



$$I = \dot{Q} \sim I_{LZ}$$

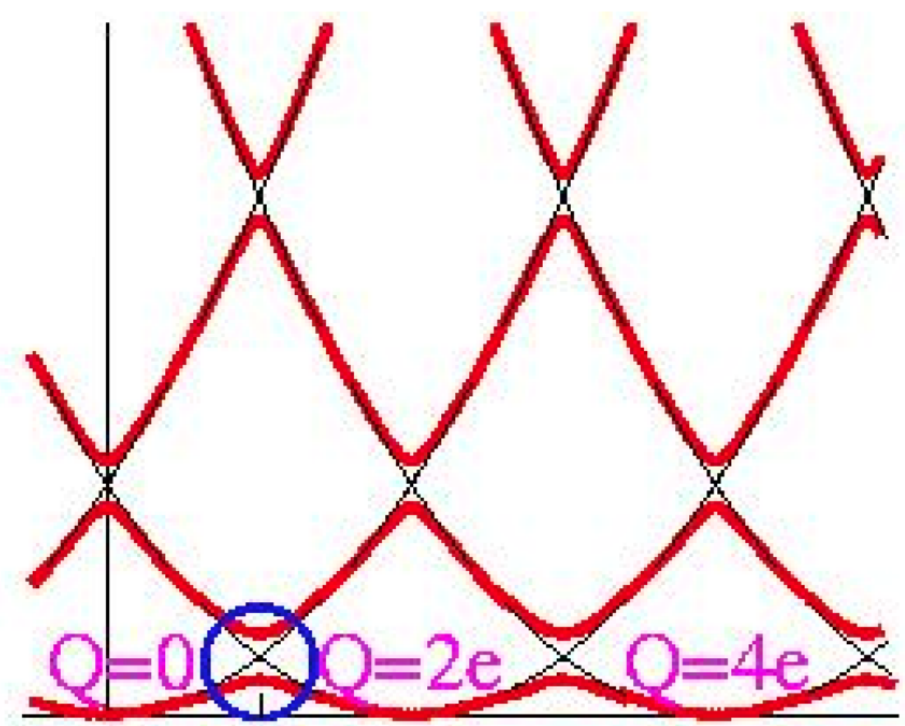
Text

$$N \sim 250$$

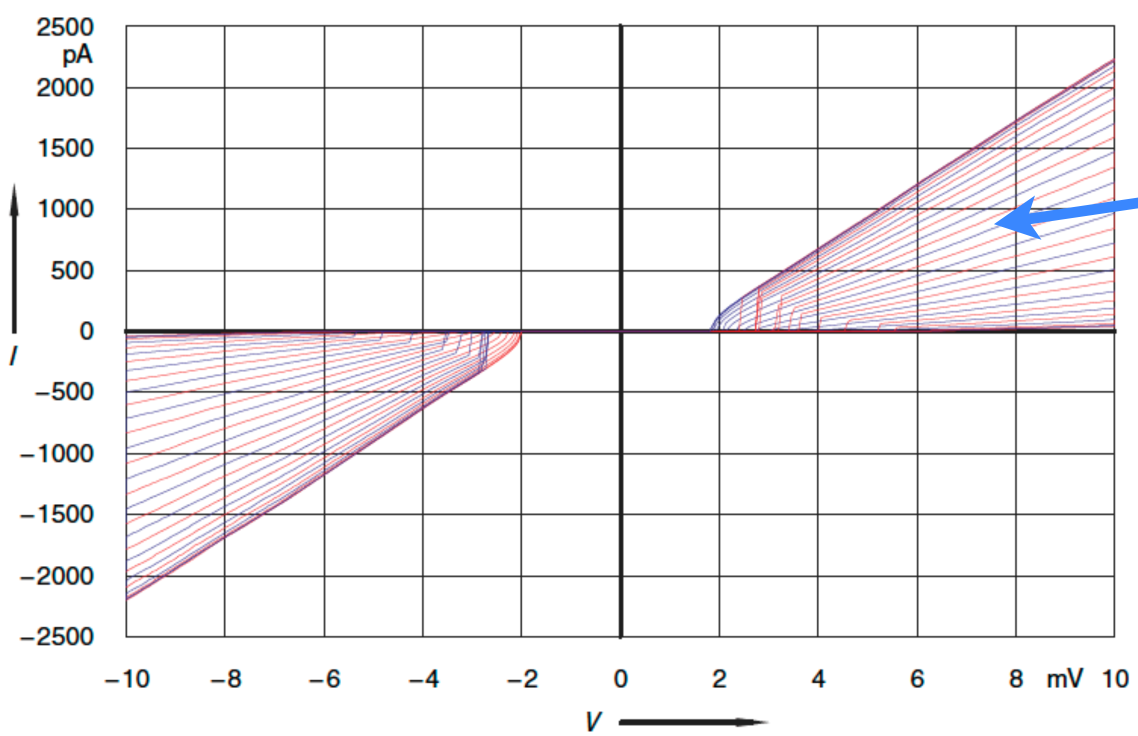
$$\frac{eV}{N} \sim 2\Delta$$

$$H(Q_n(t)) = \frac{(2em_n - Q_n(t))^2}{2C} - E_\mathrm{J} \cos\phi_n$$

$$\langle V_n \rangle = \left\langle \frac{Q_n - 2em_n}{C} \right\rangle = \frac{Q_{n+1} + Q_{n-1} - 2Q_n}{C_0}$$



Transport after depinning



$$\frac{dI}{dV} \propto [E_{\mathrm{J}}(\Phi)]^2$$

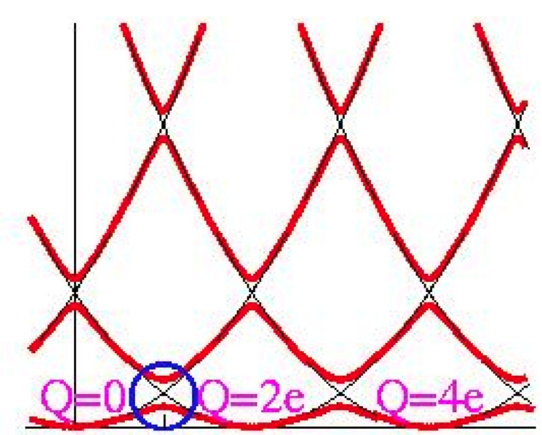
$$\frac{eV}{N} \ll 2\Delta$$

No quasiparticles ?

Two scenarios:

1) Multiple LZ in a single junction, high $V > 2\Delta/e$ on that junction, quasiparticle relaxation

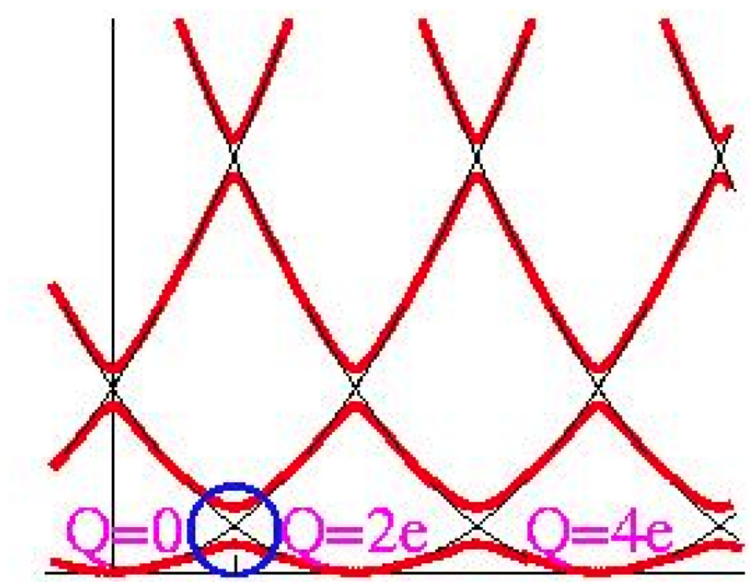
Drawback: no dependence on flux



2) Many (majority of junctions) are, e.g., in the second band, higher pinning voltage, relaxation via incoherent Cooper pair tunneling

Transport after depinning

Many (majority of junctions) are, e.g., in the second band, higher pinning voltage, relaxation via incoherent Cooper pair tunneling



$$V_{sw} \approx \frac{NE_C}{2e} \Lambda^{-\frac{2}{3}} \tilde{R}^{2/3}$$

$$\tilde{R} \equiv \frac{(\Delta U)^2}{E_C^2}$$

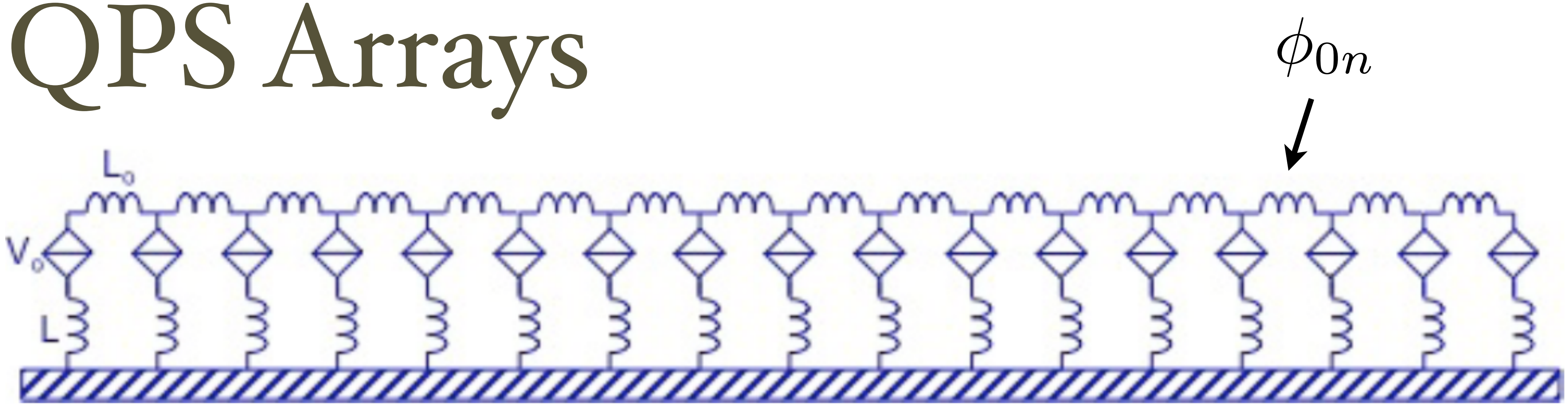
$$U[Q] = E_1(Q)$$

First Bloch band

$$\Delta U_{1Band} \gg \Delta U_{0Band}$$

$$I \propto [E_{\mathrm{J}}(\Phi)]^2$$

QPS Arrays



$$H = \frac{1}{2}\sum_{i,j} U_{i,j}\,(n_i + \delta\phi_i)(n_j + \delta\phi_j) - \sum_i E_S \cos(q_{i+1} - q_i)$$

$$U_{i,j} \approx \Lambda E_L e^{-\frac{|i-j|}{\Lambda}} \qquad \Lambda \equiv \sqrt{\frac{L}{L_0}} \gg 1 \qquad E_L \equiv \frac{\Phi_0^2}{2L}$$

$$\mathcal{L} = \sum_n \left[\frac{1}{2} C_B(\Phi_n + F_n)\dot{\Phi}_n^2 - \frac{(\Phi_n - \Phi_{n-1})^2}{2L_0} - E_0(\Phi_n + F_n)\right]$$

$$F_n = \Phi_0 \sum_{i=-\infty}^{n} \delta\phi_i \qquad \Phi_n = \Phi_0 \sum_{i=-\infty}^{n} \phi_{0i}$$

Weak (correlated) disorder

$$H = \frac{1}{2}\sum_{i,j} U_{i,j}\,(n_i + \delta\phi_i)(n_j + \delta\phi_j) - \sum_i E_S \cos(q_{i+1} - q_i)$$

$$\mathcal{L} = \sum_n \left[\frac{1}{2}C_B(\Phi_n + F_n)\dot{\Phi}_n^2 - \frac{(\Phi_n - \Phi_{n-1})^2}{2L_0} - E_0(\Phi_n + F_n)\right]$$

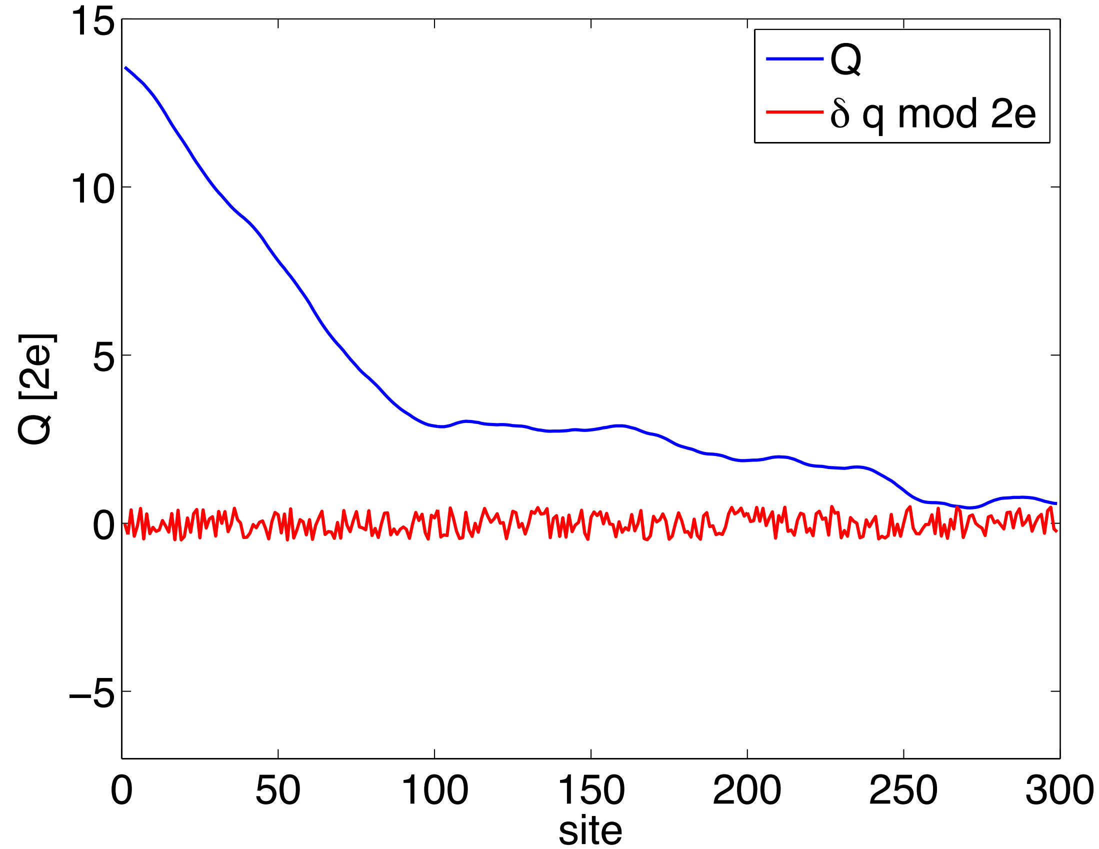
$$F_n = \Phi_0 \sum_{i=-\infty}^{n} \delta\phi_i$$

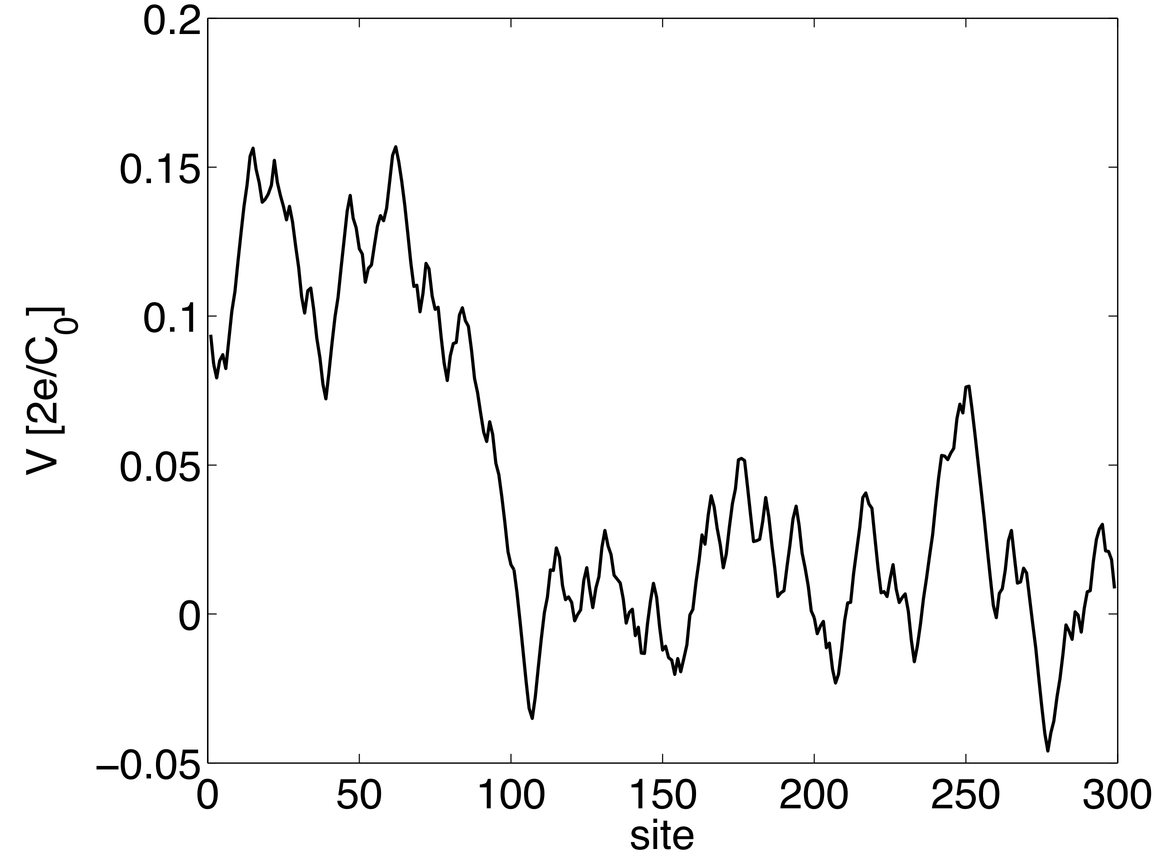
Weak disorder $\gamma \ll 1$

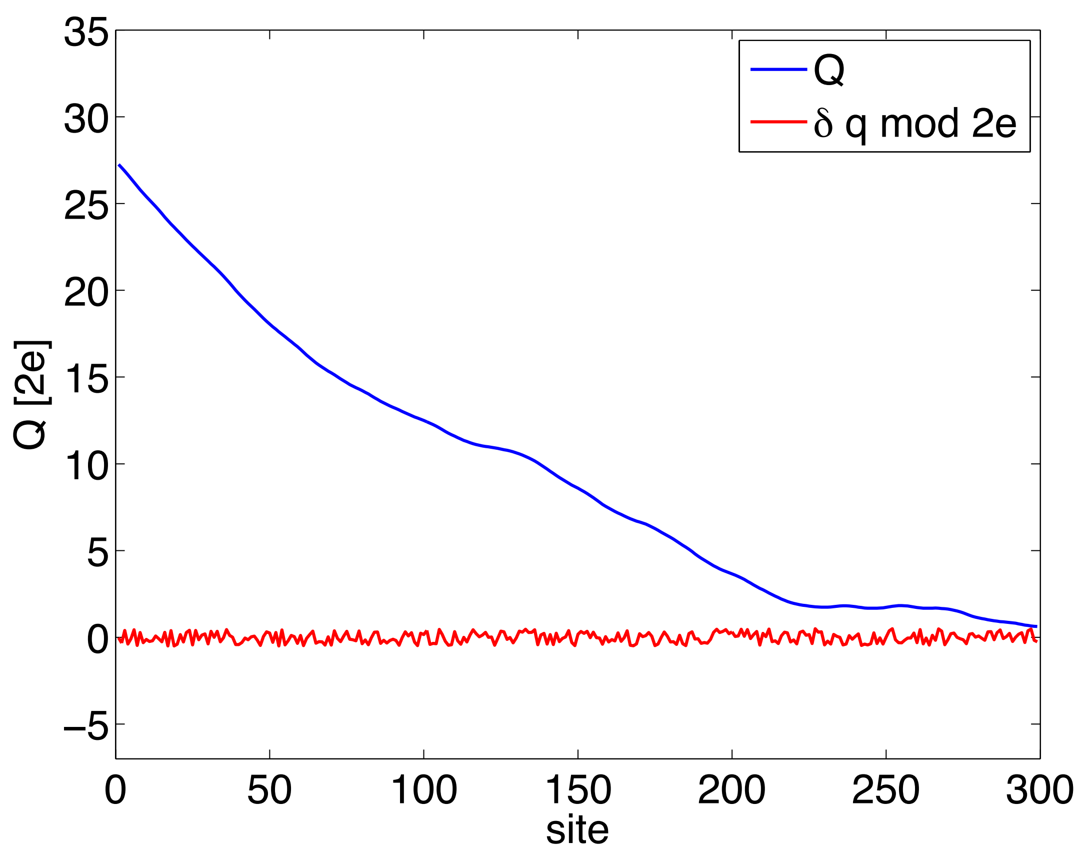
$$\delta\phi_i \in \left[-\frac{\gamma}{2}, \frac{\gamma}{2}\right] \qquad P(\delta\phi_i) = \frac{1}{\gamma}\,\theta\left(\frac{\gamma}{2} - |\delta\phi_i|\right)$$

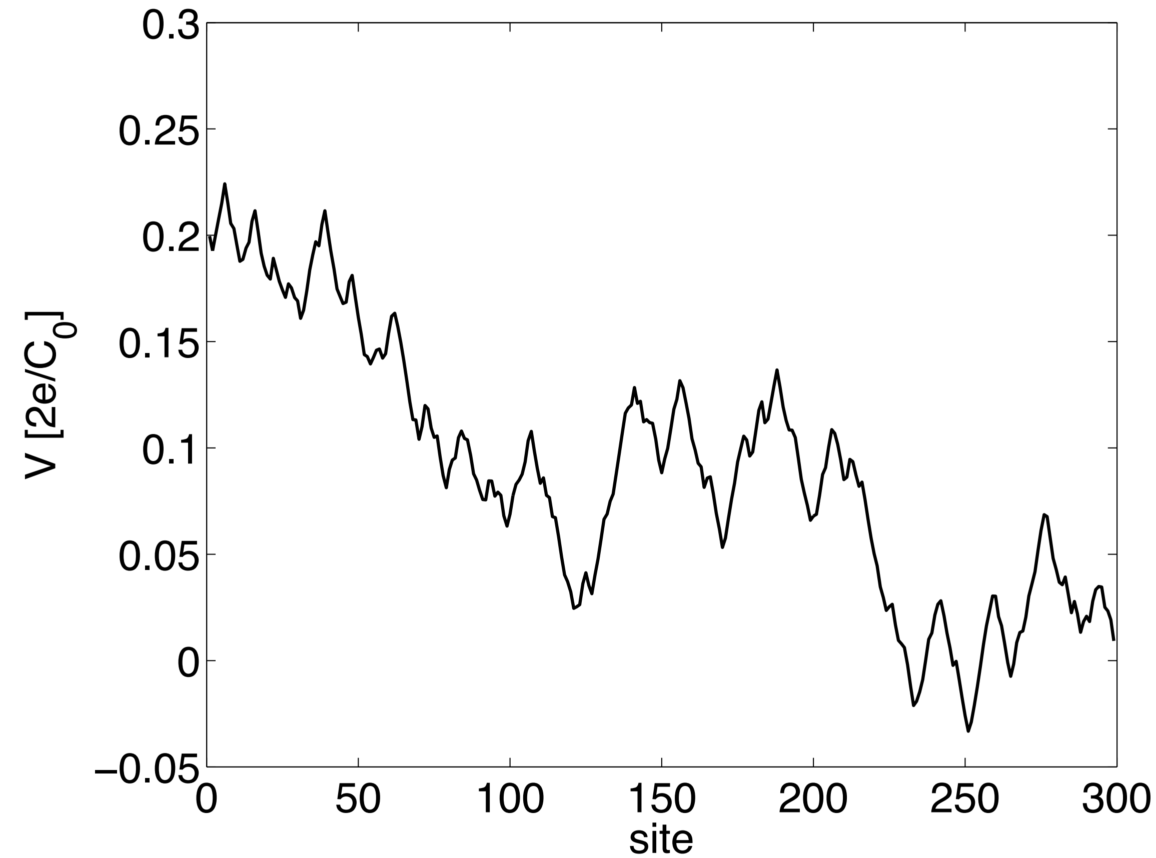
$$L_{corr} = -\frac{1}{\ln\left(\frac{\sin(\pi\gamma)}{\pi\gamma}\right)} \qquad \langle E_0\,(Q + F_i)\,E_0\,(Q + F_j)\rangle_{dis} \sim \exp\left[-\frac{|i-j|}{L_{\mathrm{corr}}}\right]$$

Numerical simulations: pinned solutions

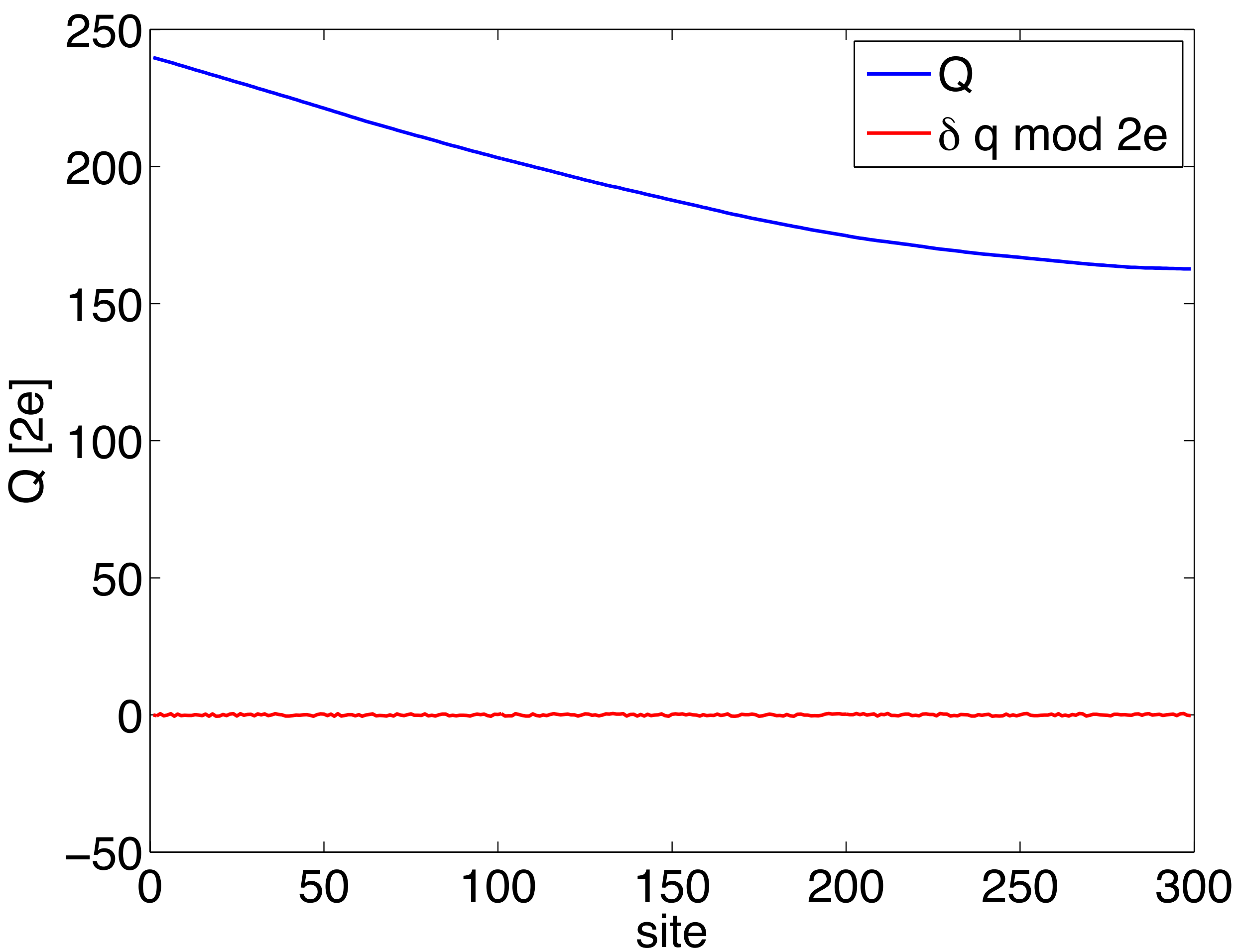


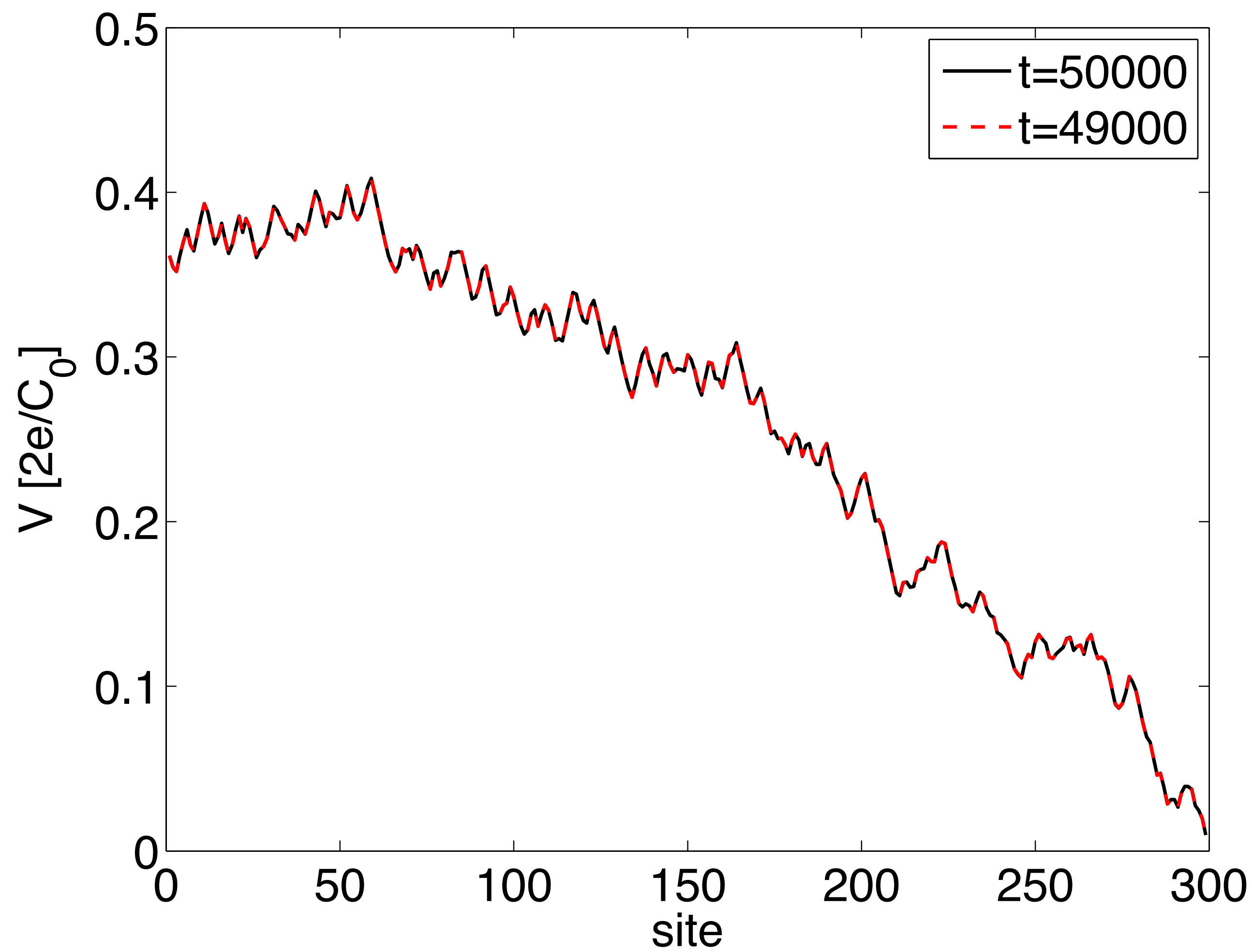




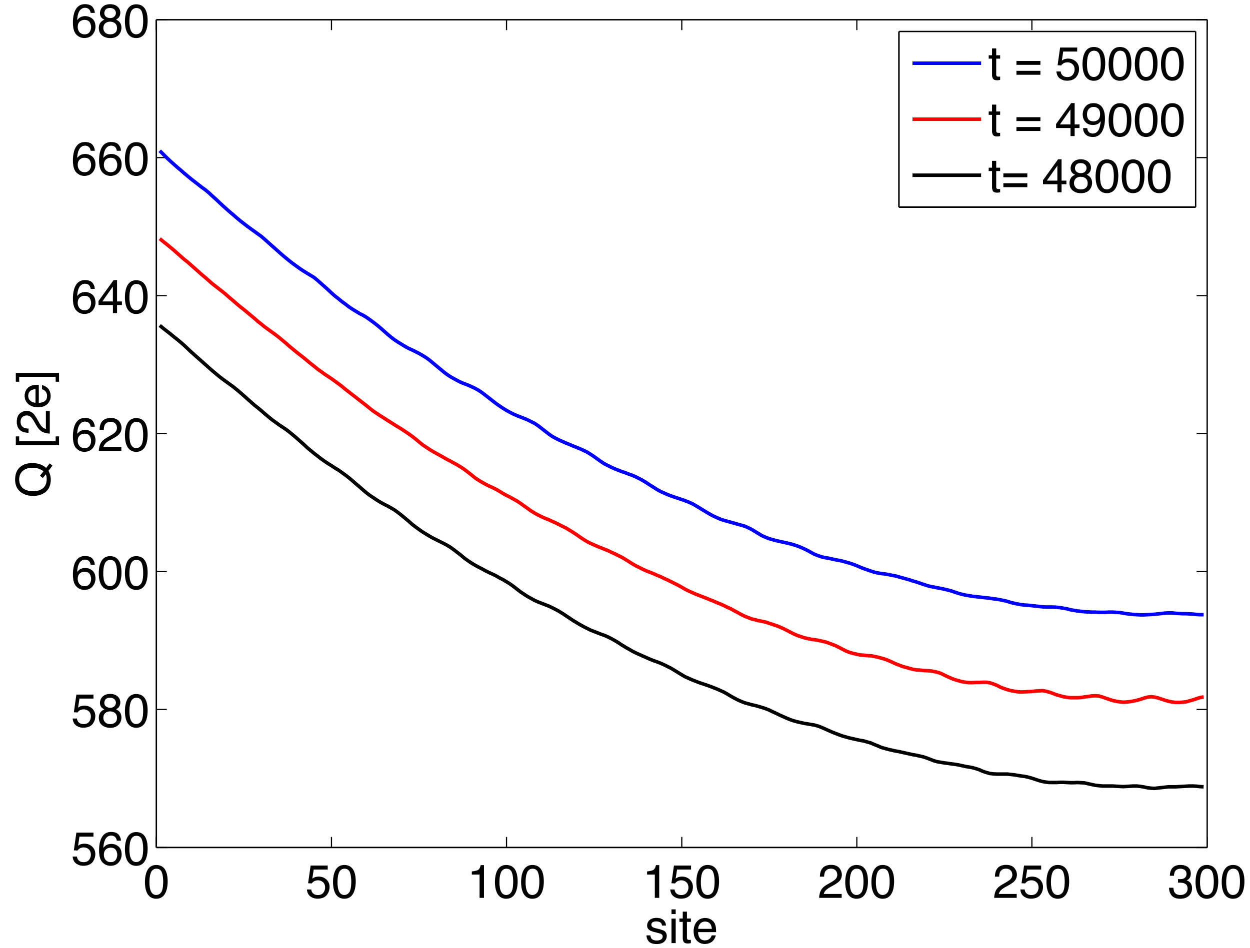


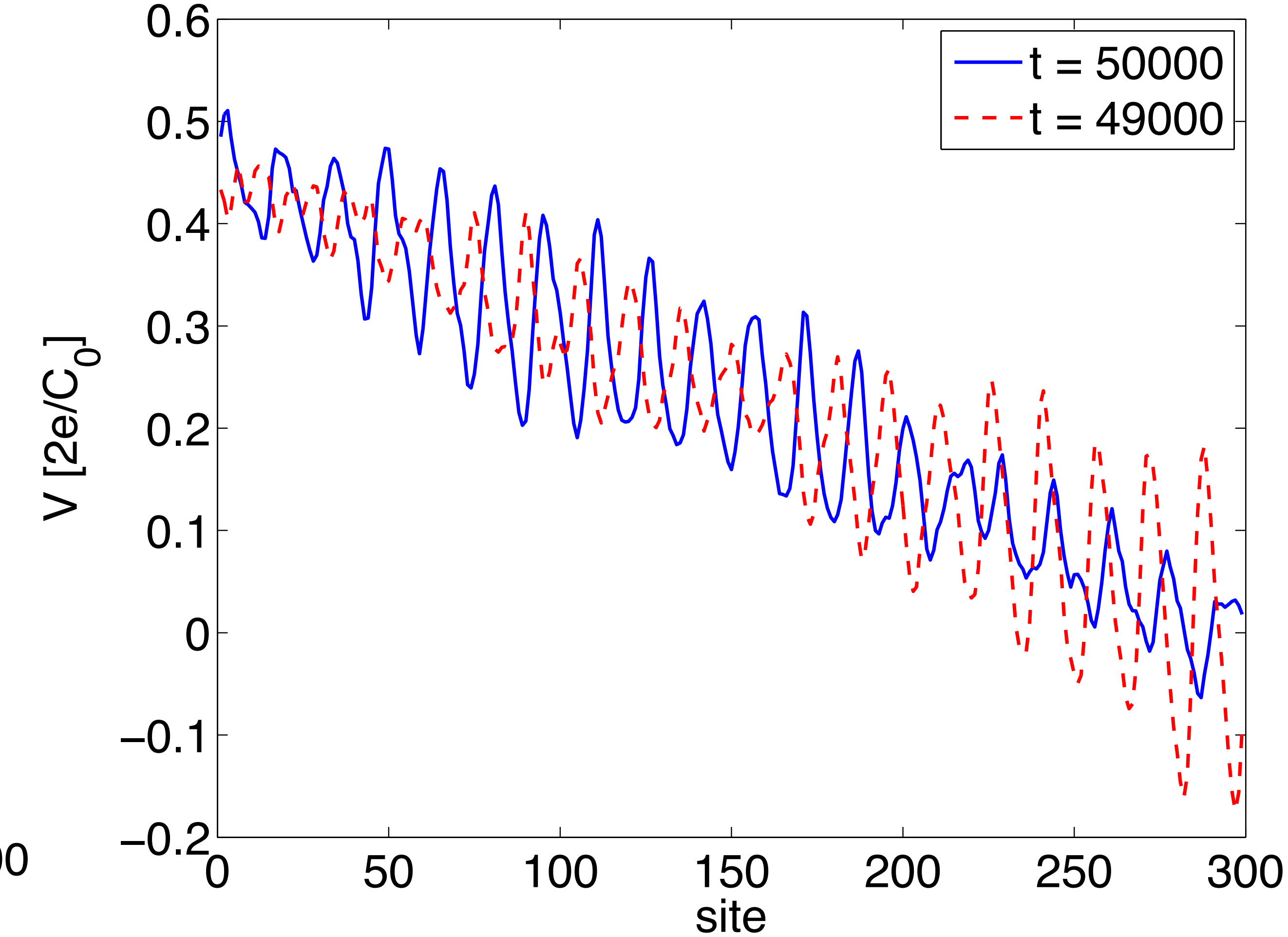
Pinned solution close to depinning



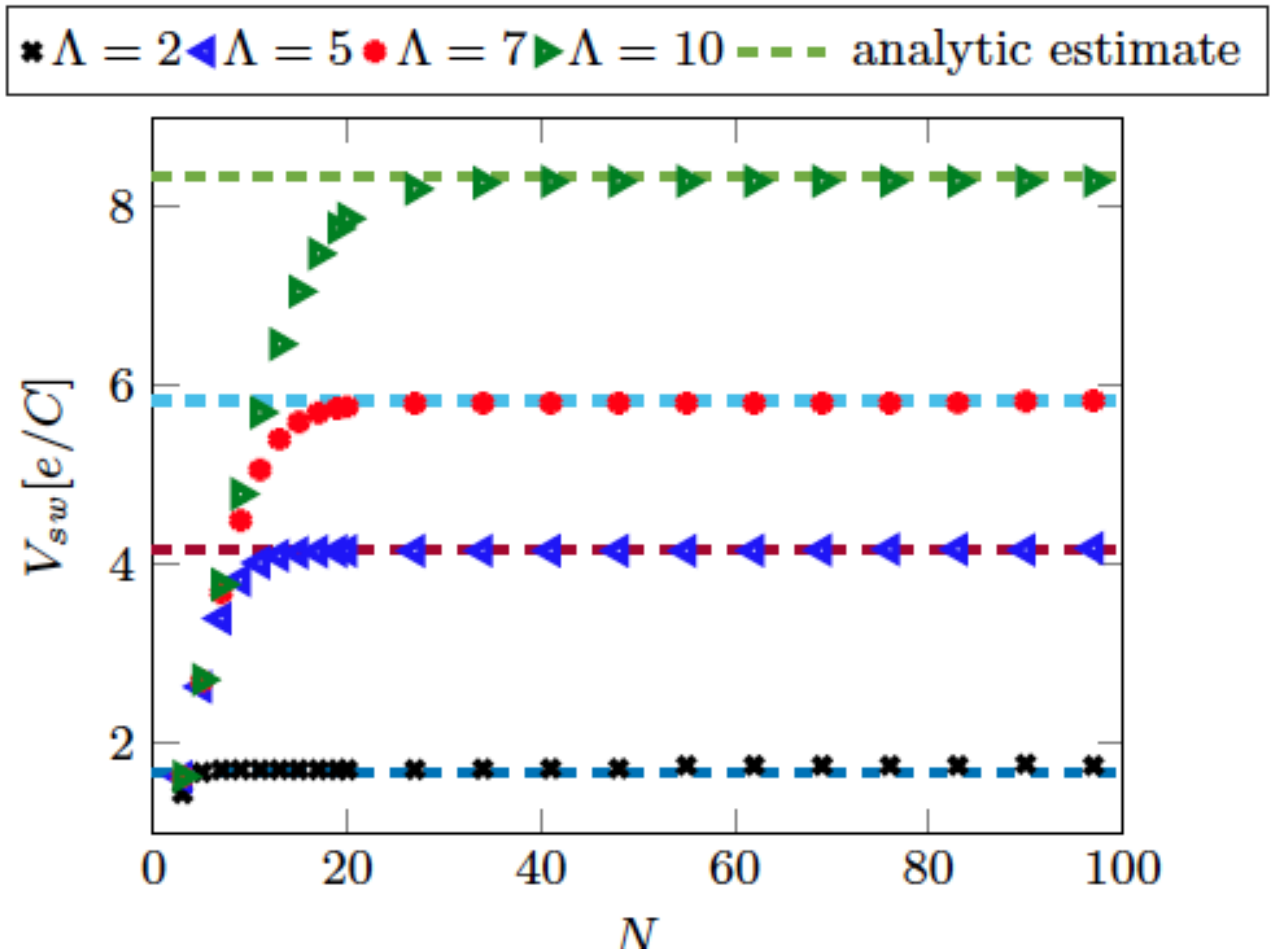


Running solution





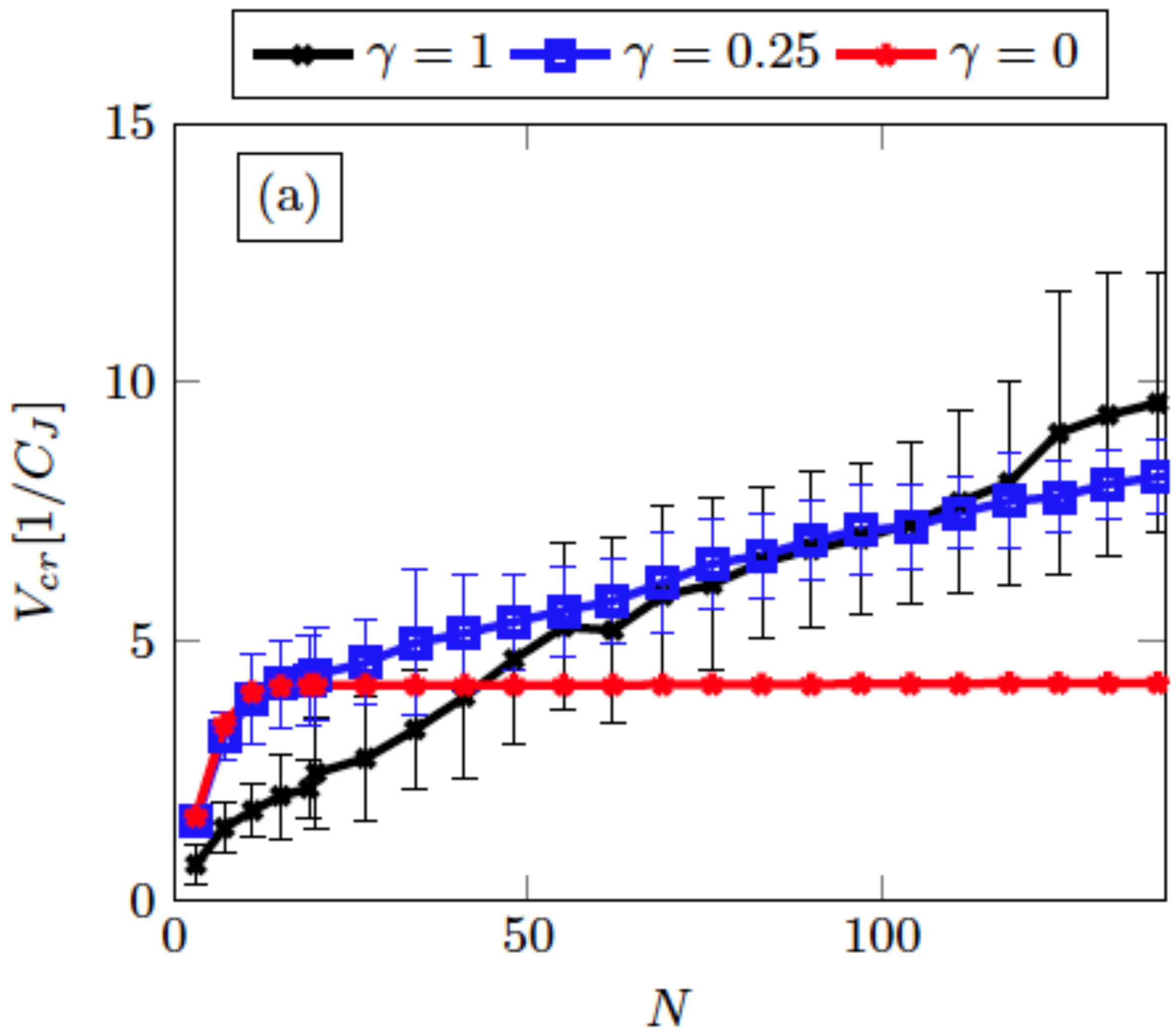
Clean limit



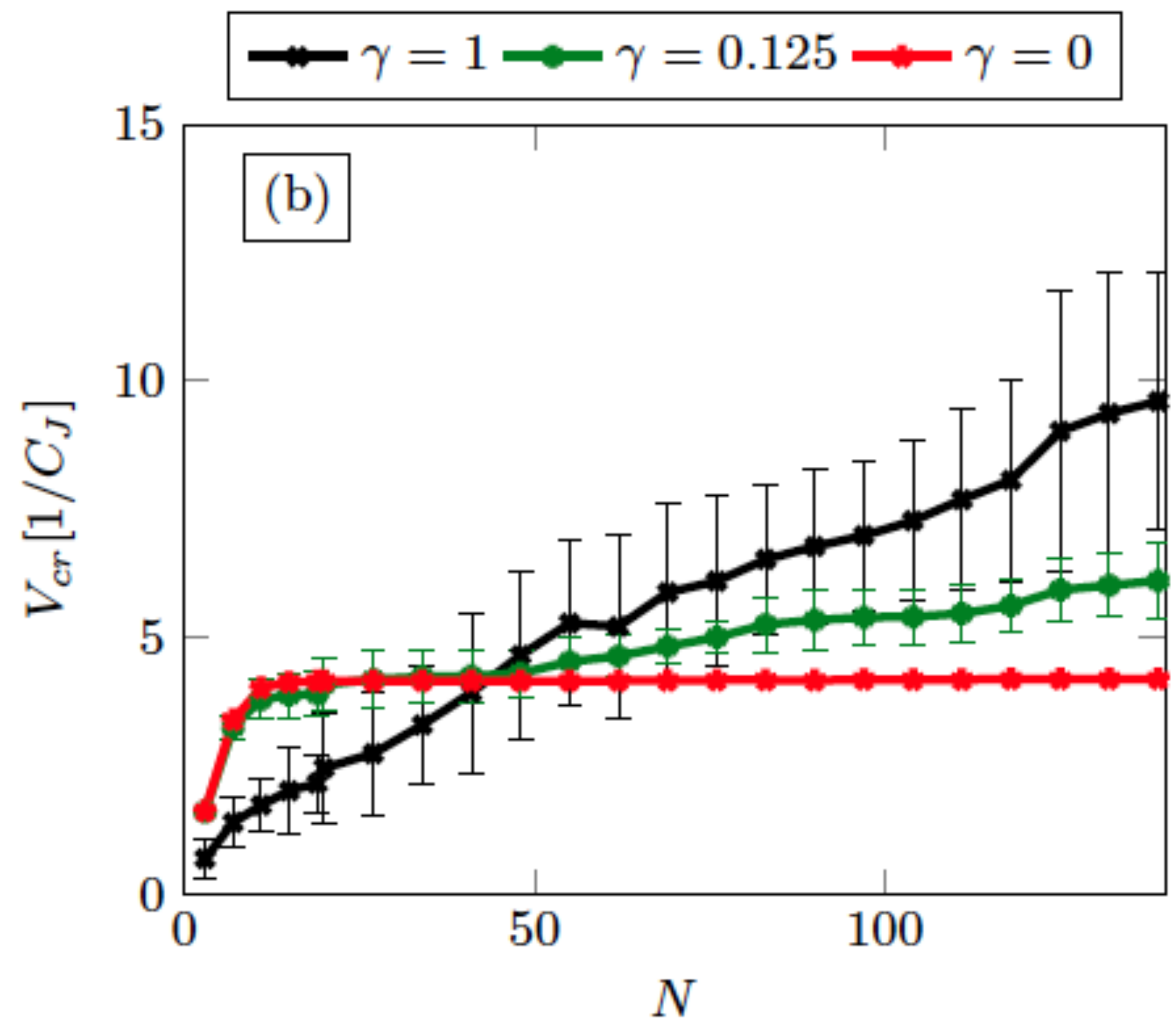
$$eV_{sw} \sim \Lambda E_C$$

Energy to create one charge soliton

Weak (correlated) disorder



$L_{\mathrm{corr}} \approx 10$



$L_{\mathrm{corr}} \approx 40$

Conclusions

1) Onset of transport: Depinning of charge density profile
2) Intermediate voltage regime: LZ + incoherent CP tunneling
3) QPS arrays: correlated weak disorder

Adiabaticity check

$Q_n \approx Q_m$ if $|n - m| < N_L$

Piece of order Larkin length is pinned and oscillates as a whole

Pinning frequency

$$\omega_{pin} \sim \sqrt{\frac{E_J E_C}{2\sqrt{N_L}}} \ll \sqrt{2E_J E_C} \qquad \text{for } E_J \sim E_C$$

adiabatic if $N_L \gg 1$

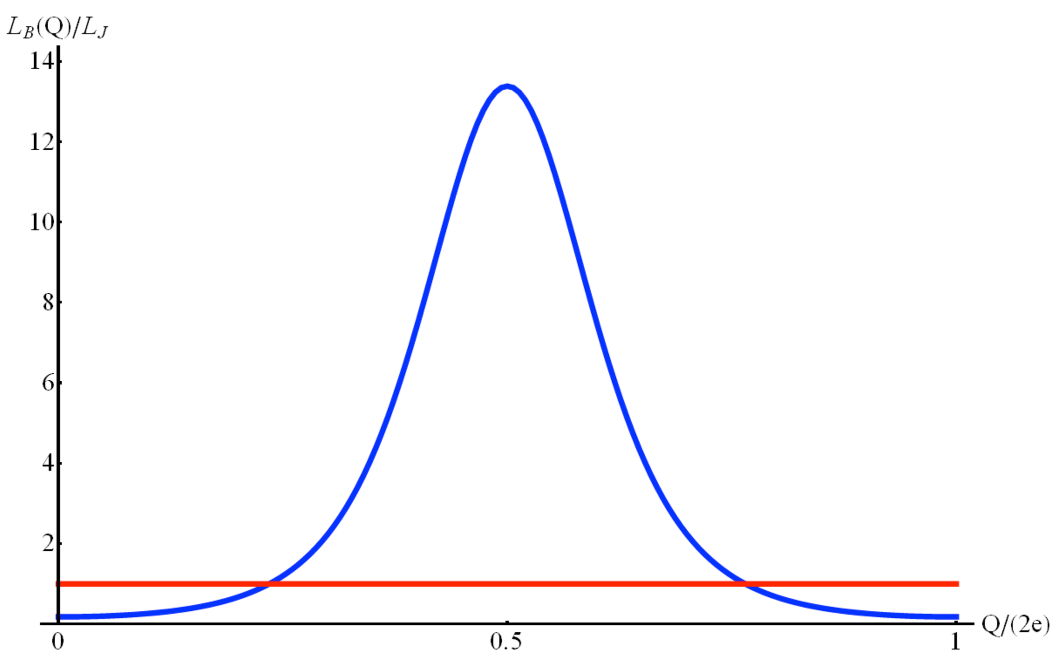
$N_L \sim \Lambda^{4/3} \tilde{R}^{-1/3}$

$\Lambda \gg 1$ helps

$$\tilde{R} \equiv \frac{(\Delta U)^2}{E_C^2}$$

Adiabaticity check

$$\mathcal{L} = \sum_n \left[\frac{1}{2} L_B(Q_n + F_n)\dot{Q}_n^2 - \frac{(Q_n - Q_{n-1})^2}{2C_0} - E_0(Q_n + F_n) \right]$$



$$E_C = 2.5 E_\mathrm{J}$$

$$Q(x) \approx Q(x') \text{ if } |x - x'| < L_L$$

inductance sampling

$$L_\mathrm{eff} \approx \frac{1}{L_L} \sum_n L_B(Q + F_n)$$

$$L_L \approx \Lambda^{4/3} \tilde{R}^{-1/3}$$

$\Lambda \gg 1$ helps