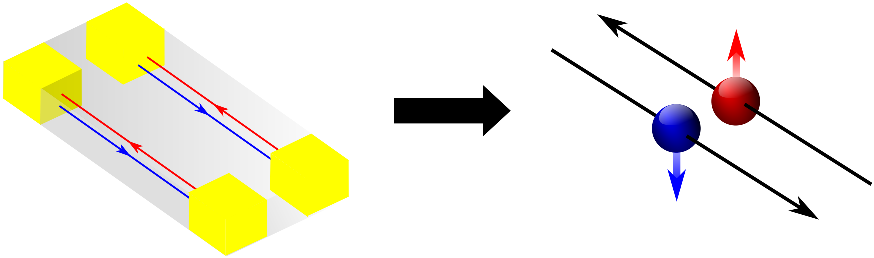
Conductivity of a generic helical liquid

Nikolaos Kainaris, **Igor Gornyi**, Sam Carr, and Alexander Mirlin | December 25, 2014

NK: KIT; IG: KIT + IOFFE PTI + LANDAU ITP; SC: KENT; AM: KIT + PNPI + LANDAU ITP



Outline

- Topological insulators and QSHE
- 1D physics and disordered Luttinger liquid
- Generic helical liquid
- Conductivity of a helical liquid

What is a topological insulator?

Topological insulator $\equiv$ bulk insulator with metallic edge/surface

Examples

- Quantum Hall effect at the plateau
- Materials with extreme spin-orbit coupling (inverted gap)
  - 2D: Quantum spin-Hall effect (HgTe/CdHgTe quantum wells)
  - 3D: Bi_xSb_{1-x}, BiTe, BiSe

Topological invariants

Topological invariants

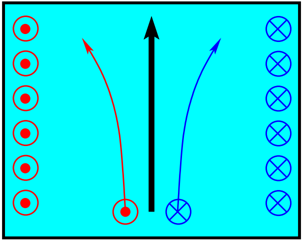
- **QHE**: time-reversal symmetry broken by magnetic field
  Chern number = # of edge states = ... $-2, -1, 0, 1, 2,$ ... ($\mathbb{Z}$)
  $\Longrightarrow$ $\mathbb{Z}$ topological insulator

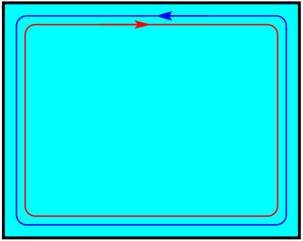
- **QSHE**: time-reversal symmetry preserved, spin-rotational broken
  band structure topological invariant ($\mathbb{Z}_2$): $n = 0$ or $n = 1$
  $\Longleftrightarrow$ odd vs. even number of Kramers pairs of edge states
  $\Longrightarrow$ $\mathbb{Z}_2$ topological insulator

Quantum spin-Hall effect

Kane, Mele ’05, Sheng et al ’05 Bernevig, Zhang ’06

No magnetic field but strong spin-orbit interaction
$\Longrightarrow$ Electrons with opposite spins feel opposite effective magnetic field



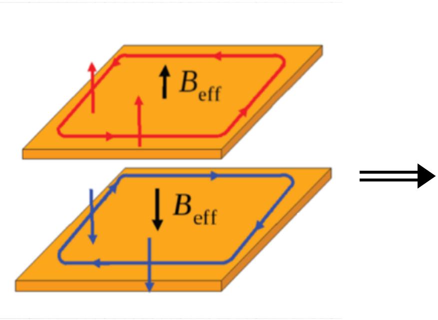


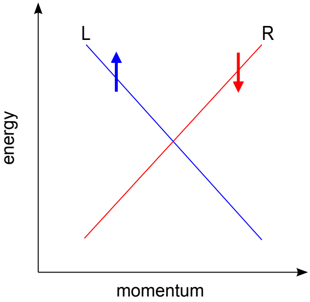
Electric current leads to spin accumulation at the edges
$\Longrightarrow$ spin-Hall effect

Extreme spin-orbit coupling opens a band gap
$\Longrightarrow$ quantum spin-Hall effect?

QSHE: Cartoon

- two QH states with opposite magnetic field
  $\rightarrow$ effective magnetic field due to spin-orbit coupling



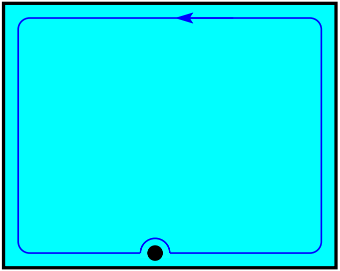


- Z_2 TI: one time reversal pair $\rightarrow$ helical edge state
- Naively: quantized conductance $G_0 = 2e^2/h$

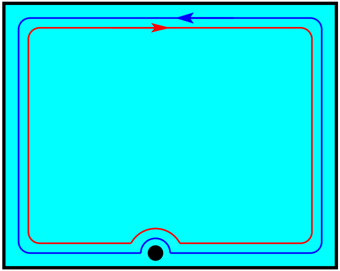
Quantum (spin-)Hall effect with disorder

Impurities do not destroy the edge (spin) current:

QHE



QSHE



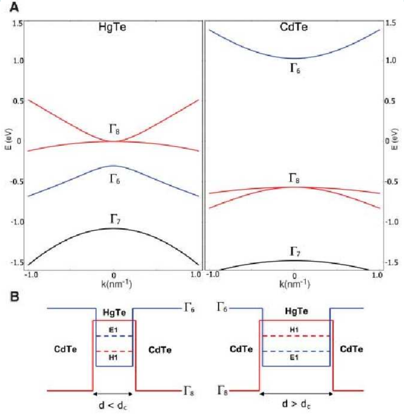
due to chirality of carriers
any disorder

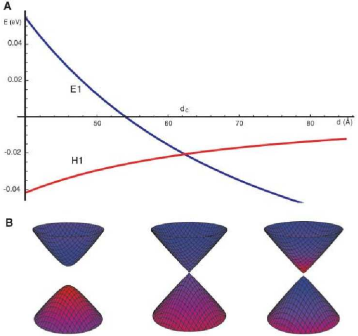
due to time-inversion symmetry
no magnetic impurities

QSHE: theory proposal

Bernevig, Hughes, Zhang ’06

HgTe/CdTe quantum well band structure





2D Dirac Hamiltonian with tunable mass: $m \gtrless 0$ when $d \lessgtr d_c$

HgTe quantum wells: Hamiltonian

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J_z-symmetric Hamiltonian in basis $E1+, H1+, E1-, H1-$
(Bernevig-Hughes-Zhang Hamiltonian):

$$H_{\text{BHZ}} = \begin{pmatrix} h(\mathbf{k}) & 0 \\ 0 & h^*(-\mathbf{k}) \end{pmatrix}, \qquad h(\mathbf{k}) = \begin{pmatrix} \epsilon(\mathbf{k}) + m(\mathbf{k}) & Ak_+ \\ Ak_- & \epsilon(\mathbf{k}) - m(\mathbf{k}) \end{pmatrix}$$

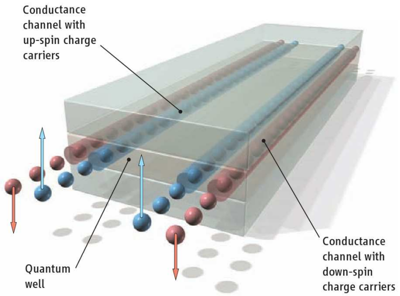
$$k_\pm = k_x \pm ik_y, \qquad \epsilon(\mathbf{k}) = C + D\mathbf{k}^2, \qquad m(\mathbf{k}) = M + B\mathbf{k}^2.$$

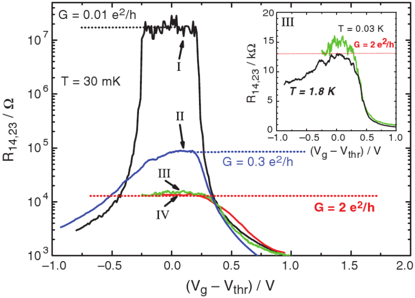
Spin-orbit interaction (block mixing) due to BIA and SIA:

$$H_{\text{SO}} = \begin{pmatrix} 0 & 0 & 2\delta_e k_+ & -\Delta_0 \\ 0 & 0 & \Delta_0 & 2\delta_h k_- \\ 2\delta_e k_- & \Delta_0 & 0 & 0 \\ -\Delta_0 & 2\delta_h k_+ & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 2ir_0 k_- & 0 \\ 0 & 0 & 0 & 0 \\ -2ir_0 k_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

QSHE: experiment

Molenkamp group '07

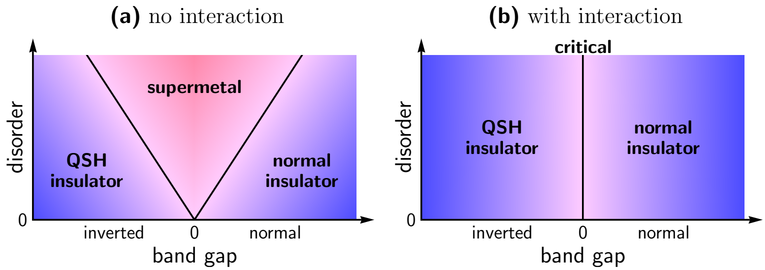




I — $d = 5.5$nm: normal insulator

II, III, IV — $d = 7.3$nm: inverted band gap — topological insulator

QSHE: Phase diagram

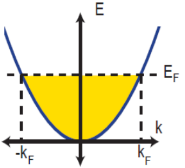


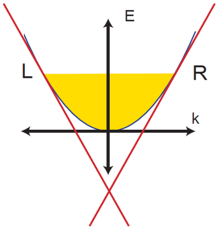
Interaction restores direct quantum spin-Hall transition via a **novel critical state**

Ostrovsky, IG, Mirlin, PRL'10

Physics in one dimension

- Many body problem, Fermi sea, Fermi surface
- 1D : Fermi surface consists of two points
  particles are moving to the right (R) or to the left (L)

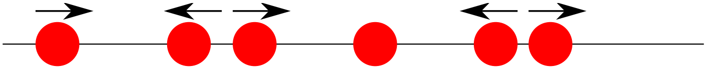




- current

$$J = e\, v_F\, (N_R - N_L)$$

Physics in 1D: strong correlations



- Electrons in 1D have no way “around” each other
- Arbitrarily weak interaction qualitatively changes the ground state
- Fundamental excitations: collective density waves - plasmons

Tomonaga-Luttinger liquid

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- Interacting model (quartic in fermions):

$$\hat{\mathcal{H}}_0 = - iv_F \int \mathrm{d}x \sum_{\eta=R,L} \eta \hat{\psi}^\dagger_\eta(x) \partial_x \hat{\psi}_\eta(x)$$

$$\hat{\mathcal{H}}_{\mathrm{int}} = g \int \mathrm{d}x \, \hat{\rho}(x)\hat{\rho}(x), \quad \rho(x) = \sum_\eta \hat{\psi}^\dagger_\eta(x)\hat{\psi}_\eta(x)$$

- Bosonization : Exact mapping of fermionic to bosonic Hilbert space

$$\hat{\psi}_\eta(x) = (2\pi\alpha)^{-1/2} \exp\left(-i\sqrt{\pi}\left[\hat{\varphi} - \eta\hat{\theta}\right]\right)$$

- Nonlocal commutation relations: $[\hat{\varphi}(x), \hat{\theta}(y)] = i\Theta(x-y)$

Operator Bosonization

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- Interacting model remains quadratic in bosons

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{\text{int}} = \frac{u}{2}\left[K(\partial_x\hat{\theta})^2 + K^{-1}(\partial_x\hat{\varphi})^2\right]$$

- characterized by two parameters

Luttinger liquid parameter

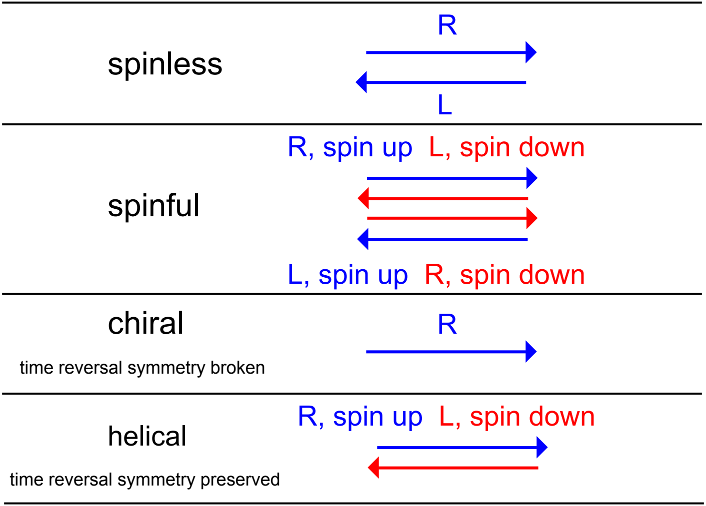
$$K = (1 + g/\pi v_F)^{-1/2}$$

Plasmon velocity

$$u = v_F/K$$

- Luttinger liquid parameter describes fermionic interaction strength: $K = 1$ noninteracting, $K < 1$ repulsive, $K > 1$ attractive

Different 1D quantum liquids



1D quantum liquids

- Luttinger liquids with these structures of modes?
- Conductivity?

Disordered Luttinger liquid

Giamarchi & Schulz '88; IG, Mirlin, Polyakov '05, '07

- Single-channel infinite wire: right(left) movers $\psi_\eta, \quad \eta = \pm$
- Spinless (spin-polarized, $\sigma = +$) or spinful ($\sigma = \pm$) electrons
- Linear dispersion, $\quad \epsilon_k = k v_F$
- Short-range weak e-e interaction, $\quad \alpha \equiv V(0)/2\pi v_F \ll 1$
- No e-e backscattering; $\quad g$-ology with g_2 and g_4
- White-noise weak ($E_F \tau_0 \gg 1$) disorder,

$$\langle U(x) U(x') \rangle = \delta(x - x')/2\pi\nu_0\tau_0 .$$

Bosonization and disorder averaging

Giamarchi & Schulz '88

- Bosonization: given realization of disorder,
- Disorder averaging. Quenched disorder: replicas ϕ_n

Bosonized replicated action (no spin):

$$S[\phi] = \frac{1}{2\pi v_F} \sum_n \int dx\, d\tau \, \{ [\partial_\tau \phi_n(x,\tau)]^2 - u^2 [\partial_x \phi_n(x,\tau)]^2 \} - \frac{v_F k_F^2}{\pi^2 \tau_0} \sum_{n,m} \int dx\, d\tau\, d\tau' \cos[2\phi_n(x,\tau) - 2\phi_m(x,\tau')]$$

Two steps: virtual & real processes

IG, Mirlin, Polyakov '05, '07; Bagrets, IG, Mirlin, Polyakov '09

- **Step 1:** Integrate out $T < \epsilon < E_F$ (RG, virtual processes)
  Giamarchi & Schulz '88 $\rightarrow$ T-dependent static disorder

$$\tau(T) = \tau_0 (T/E_F)^{2\alpha},$$

  all power-law (Luttinger) terms $\propto (E_F/T)^{\gamma}$ in renormalized couplings

- **Step 2:** Refermionize
  solve kinetic equation: classical (Drude) conductivity

$$\sigma_D(T) \propto \tau(T) \propto T^{2\alpha},$$

  inelastic relaxation processes;
  interference and dephasing

1D vs. 2D

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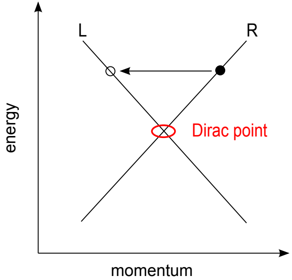
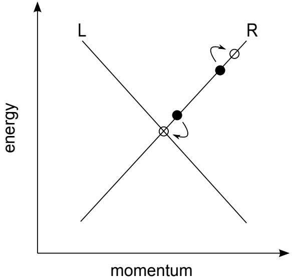
- Luttinger ("non-Fermi") liquid $\longleftrightarrow$ Zero-bias anomaly in TDOS
  $\nu(\epsilon)/\nu_0 \sim (\epsilon/E_F)^\gamma \ll 1$ $\qquad$ $\nu(\epsilon)/\nu_0 \sim \exp[-\frac{1}{8\pi^2 g}\ln^2\epsilon] \ll 1$
- Giamarchi-Schulz RG $\longleftrightarrow$ Finkel'stein RG
- Renormalization of disorder $\longleftrightarrow$ T-dependent screening
- Anderson localization $\longleftrightarrow$ Strong WL-corrections ($L_\varphi \sim \xi$)

Helical liquid

- Linear spectrum with Dirac point
- Elastic backscattering from nonmagnetic impurity forbidden
- Inelastic (two-particle) backscattering allowed

Xu & Moore '06, Wu, Bernevig, Zhang '06

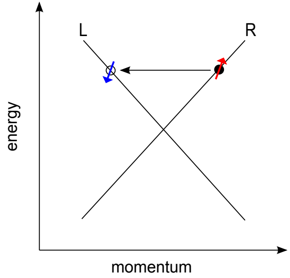


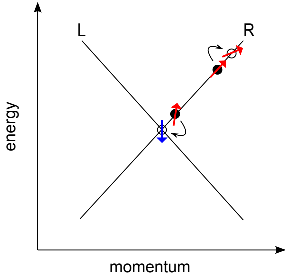


Generic helical liquid

- Linear spectrum with Dirac point
- Elastic backscattering from nonmagnetic impurity forbidden
- Broken S_z symmetry: Inelastic backscattering allowed

Schmidt, Rachel, von Oppen, Glazman ’12





Motivation

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Goal

Theory of transport in a generic helical liquid.

Previous work

- Correction to conductance of a short edge for weakly interacting electrons[1,2]
- Luttinger liquid renormalization[2,3,4,5]

This work

- Conductivity of long edge channels including LL renormalization

[1] T. L. Schmidt, S. Rachel, F. von Oppen and L. I. Glazman, Phys. Rev. Lett. **108**, 156402 (2012).
[2] F. Crépin, J. C. Budich, F. Dolcini, P. Recher, and B. Trauzettel, Phys. Rev. B **86**, 121106(R) (2012).
[3] A. Ström, H. Johannesson, and G. I. Japaridze, Phys. Rev. Lett. **104**, 256804 (2010).
[4] N. Lezmy, Y. Oreg, and M. Berkooz, Phys. Rev. B **85**, 235304 (2012).
[5] F. Geissler, F. Crépin, and B. Trauzettel, Phys. Rev. B **89**, 235136 (2014)

Model

Generic helical liquid

with SO interaction – block mixing = S_z broken, TR respected.
Rotation of basis (spins vs. chirality/helicity) [1]

$$\begin{pmatrix}\psi_{k,\uparrow}\\ \psi_{k,\downarrow}\end{pmatrix} = B_k \begin{pmatrix}\psi_{k,R}\\ \psi_{k,L}\end{pmatrix}, \qquad B_k^\dagger B_k = \mathbb{1}, \qquad B_k = B_{-k}$$

General form of B_k for $k \ll k_0$ (here k_0^{-1} – strength of SO interaction):

$$B_k = \begin{pmatrix} 1-\frac{k^4}{2k_0^4} & -\frac{k^2}{k_0^2} \\ \frac{k^2}{k_0^2} & 1-\frac{k^4}{2k_0^4}\end{pmatrix}, \qquad \left[B_k^\dagger B_p\right]_{\eta,\eta'} = \delta_{\eta,\eta'} + \eta\, \delta_{\bar\eta,\eta'} \frac{k^2-p^2}{k_0^2},$$

[1] Schmidt, Rachel, von Oppen, Glazman (PRL 2012)

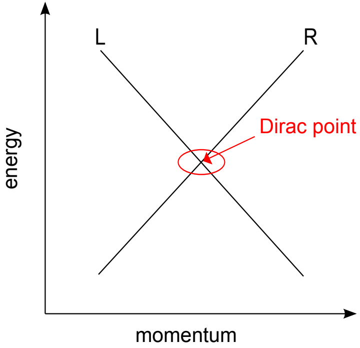
Model

Microscopic model for 1D time reversal invariant quantum liquid[2]

Spinless Fermions with momentum k and chirality η:

$$\hat{\mathcal{H}}_0 = v_F \sum_{k,\eta} \eta\, k \;\; \hat{\psi}^\dagger_{\eta,k} \hat{\psi}_{\eta,k}$$

Linear spectrum



[2] Schmidt, Rachel, von Oppen, Glazman (PRL 2012)

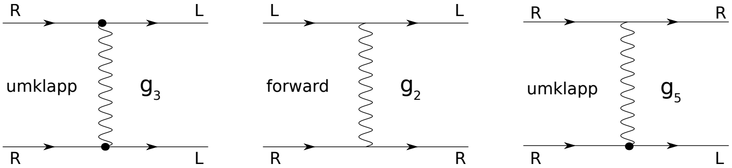
Model

screened two-particle interaction

$$\hat{\mathcal{H}}_2 = \frac{V}{L}\sum_{k,p,q,\eta} \hat{\psi}^\dagger_{\eta,k}\hat{\psi}^\dagger_{\bar{\eta},p}\hat{\psi}_{\bar{\eta},p+q}\hat{\psi}_{\eta,k-q}$$

$$\hat{\mathcal{H}}_3 = \frac{V}{k_0^4 L}\sum_{k,p,q,\eta} \left(k^2-(k-q)^2\right)\left(p^2-(p+q)^2\right)\hat{\psi}^\dagger_{\eta,k}\hat{\psi}^\dagger_{\eta,p}\hat{\psi}_{\bar{\eta},p+q}\hat{\psi}_{\bar{\eta},k-q}$$

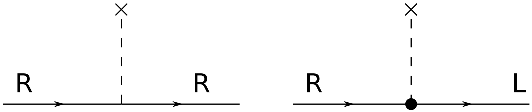
$$\hat{\mathcal{H}}_5 = -\frac{V}{k_0^2 L}\sum_{k,p,q,\eta} \eta(k^2-p^2)\hat{\psi}^\dagger_{\eta,k+q}\hat{\psi}^\dagger_{\bar{\eta},p-q}\hat{\psi}_{\eta,p}\hat{\psi}_{\eta,k} + h.c.$$



Model: disorder

Short range, non-magnetic impurities

$$\hat{\mathcal{H}}_{\text{imp}} = \frac{U}{L} \sum_{k,p,\eta} \left(\hat{\psi}^{\dagger}_{\eta,k} \hat{\psi}_{\eta,p} + \eta \frac{k^2 - p^2}{k_0^2} \hat{\psi}^{\dagger}_{\eta,k} \hat{\psi}_{\bar{\eta},p} \right)$$



Magnetic impurities & conducting islands (Kondo, spin glass...):

J. Maciejko, C. Liu, Y. Oreg, X.-L. Qi, C. Wu, and S.-C. Zhang '09

V. Cheianov, L. Glazman '13; J.I. Väyrynen, M. Goldstein, L.I. Glazman '13; J.I. Väyrynen, M. Goldstein, Y. Gefen, L.I. Glazman '14

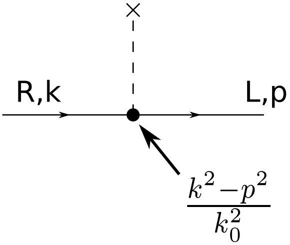
B.L. Altshuler, I.L. Aleiner, V.I. Yudson '13

Not included here; might be important for experiments

Model

Microscopic model for 1D TR invariant quantum liquid

- momentum factor ensures TRI
- k_0^{-1} measures spin-orbit coupling



Methods

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1. $\ell_\phi \ll \ell$: Semiclassical conductivity

Kinetic equation

- Weakly interacting electrons $K \simeq 1$
- Classical electric field $\omega \ll T$
- *ac* ($\omega \gg \tau^{-1}$) and *dc* ($\omega \ll \tau^{-1}$) conductivity

Bosonization + Kubo formula

- Arbitrary interactions $K \in (0, 1]$
- Classical electric field $\omega \ll T$ and "photons" $\omega \gg T$
- *ac* conductivity

2. $\ell_\phi \gg \ell$: Quantum interference effects

- mapping[3] to Giamarchi-Schulz model

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[3] Ström, Johanneson, Japaridze (2010)

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Dominant scattering mechanisms

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- process has to change chirality of incoming particles

$$j = e\, v_F\, (N_R - N_L)$$

- Disorder backscattering reduced by TRI
$\Rightarrow$ forward scattering dominant in combined processes
- Transport: combined effects of interaction and disorder

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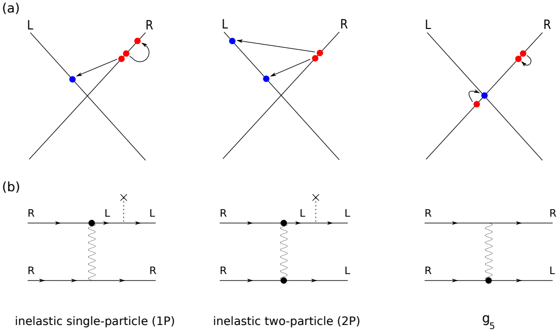

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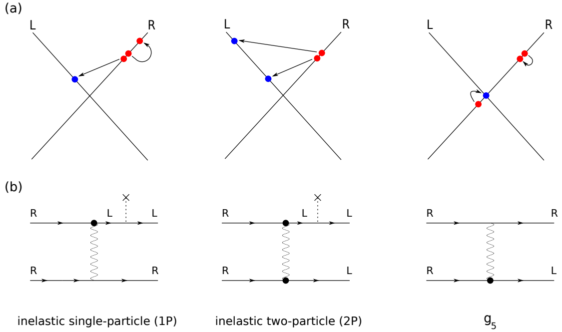

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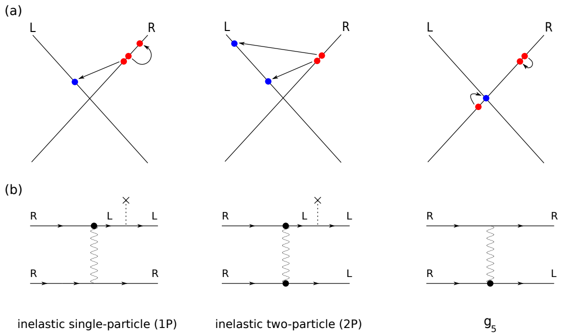

Clean case: *ac* vs *dc* conductivity

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Scattering time τ in presence of g_5 interaction

	$T \ll v_F k_F$	$T \gg v_F k_F$
τ_{ac}	$0.16\left(\frac{v_F}{V}\right)^2\left(\frac{k_0}{k_F}\right)^4\frac{T}{(v_F k_F)^2}e^{\frac{v_F k_F}{T}}$	$6.5\times 10^{-3}\left(\frac{v_F}{V}\right)^2\frac{1}{v_F k_0}\left(\frac{v_F k_0}{T}\right)^5$
τ_{dc}	$0.81\left(\frac{v_F}{V}\right)^2\left(\frac{k_0}{k_F}\right)^4\frac{1}{v_F k_F}e^{\frac{v_F k_F}{T}}$	$0.014\left(\frac{v_F}{V}\right)^2\frac{1}{v_F k_0}\left(\frac{v_F k_0}{T}\right)^5$

- $T \gg v_F k_F$: τ independent of ω
  $\Rightarrow$ Interpolation by Drude's law :

$$\sigma(\omega) \simeq \frac{2e^2 v_F}{h}\frac{1}{\tau^{-1} - i\omega}$$

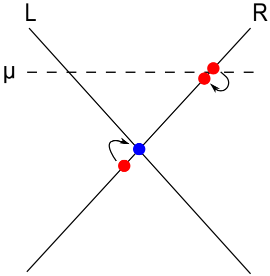
- $T \ll v_F k_F$: Non-Drude

Clean case: Discussion

Scattering time in presence of g_5 interaction at $T \ll k_F$

- Kinetics:
  one particle at Dirac point
- thermally activated at low temperatures
- parametrical suppresion at $\omega = 0$:

$$\tau_{dc} = \frac{E_F}{T} \tau_{ac} \gg \tau_{ac}$$



Disordered case: Effective action

Bosonization and average over white noise disorder leads to effective action

$$S_{2\mathrm{P}} = -\, g_{2\mathrm{P}} \sum_{a,b} \int \mathrm{d}x \mathrm{d}\tau \mathrm{d}\tau' \, \cos\left\{ 2\sqrt{4\pi} \left[\varphi_a(x,\tau) - \varphi_b(x,\tau') \right] \right\},$$

$$S_{1P} = -\, g_{1\mathrm{P},1} \sum_{a,b} \int \mathrm{d}x \mathrm{d}\tau \mathrm{d}\tau' \, \partial_x^2 \theta_a(x,\tau) \partial_x^2 \theta_b(x,\tau') \cos\left\{ \sqrt{4\pi} \left[\varphi_a(x,\tau) - \varphi_b(x,\tau') \right] \right\}$$

$$+\, g_{1\mathrm{P},2} \sum_{a,b} \int \mathrm{d}x \mathrm{d}\tau \mathrm{d}\tau' \, \partial_x^2 \theta_a(x,\tau) \partial_x \theta_b(x,\tau') \sin\left\{ \sqrt{4\pi} \left[\varphi_a(x,\tau) - \varphi_b(x,\tau') \right] \right\},$$

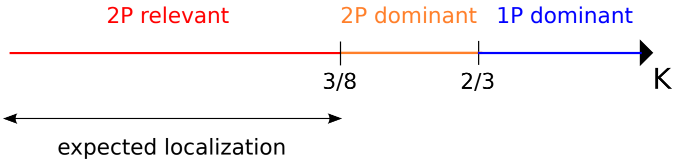
$$S_R = -\, g_{\mathrm{R}} \sum_{a,b} \int \mathrm{d}x \mathrm{d}\tau \mathrm{d}\tau' \, \partial_x \theta_a(x,\tau) \partial_x \theta_b(x,\tau') \cos\left\{ \sqrt{4\pi} \left[\varphi_a(x,\tau) - \varphi_b(x,\tau') \right] \right\}.$$

with coupling constants

$$g_{1\mathrm{P}} \propto D_f g_5^2, \qquad g_{2\mathrm{P}} \propto D_f g_3^2, \qquad g_{\mathrm{R}} \propto D_b \tag{1}$$

Disordered case

Phase diagram of conductivity:

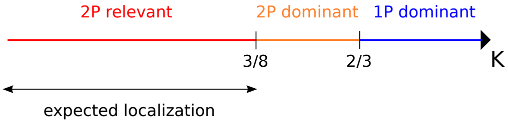


Transport in the presence of disorder

- Drude's law for conductivity as a function of ω:

$$\sigma = \frac{2e^2 u}{h} \frac{1}{\tau^{-1} - i\omega}$$

Results: LL renormalization



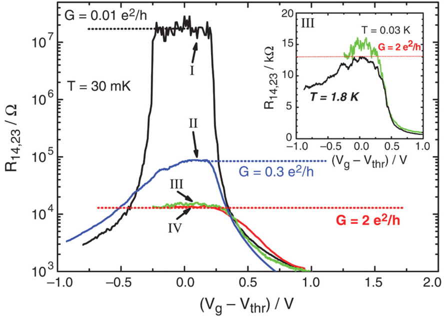
- crossover between scattering mechanisms at $K = 2/3$

$$\tau_{2P}^{-1} \sim \left(\frac{V}{u}\right)^2 \frac{D_f}{u} \left(\frac{k_F}{k_0}\right)^2 \frac{1}{(ak_0)^6} \left(\frac{a \max(\omega, T)}{u}\right)^{8K-2},$$

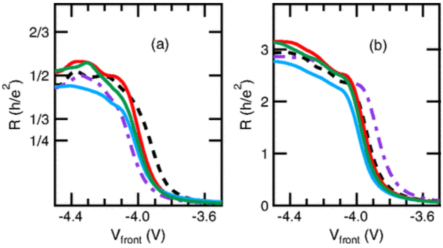
$$\tau_{1P}^{-1} \sim \left(\frac{V}{u}\right)^2 \frac{D_f}{u} \frac{1}{(ak_0)^4} \left(\frac{a \max(\omega, T)}{u}\right)^{2K+2}.$$

- backscattering has no effect to first order in D_b.

Experiment



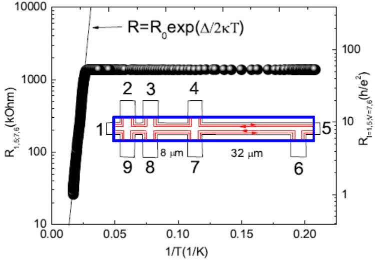
König et al. (Science 2007)

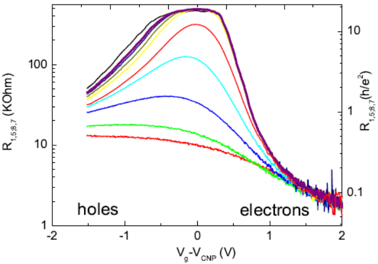


Knez et al. (PRL 2014)

- HgTe/CdTe: short ($\sim 1\mu m$) edges
- InAs/GaSb: longer ($\sim 10\mu m$) edges

Experiment





Gusev, Kvon, *et al.* '12, '13:

long edges (5-50 μm),

resistance much higher than quantum resistance,

temperature dependence saturates

Summary

- Behavior of conductivity:
  - disordered case: Drude-like
  - clean case: Non-Drude at $k_F v_F \gg T$
- Combination of interaction and forward scattering off disorder: temperature or frequency dependent σ
- LL effects cause renormalization of exponents as a function of K

N. Kainaris, I. V. Gornyi, S. T. Carr, and A. D. Mirlin,
"Conductivity of a generic helical liquid",
Phys. Rev. B. **90**, 075118 (2014)